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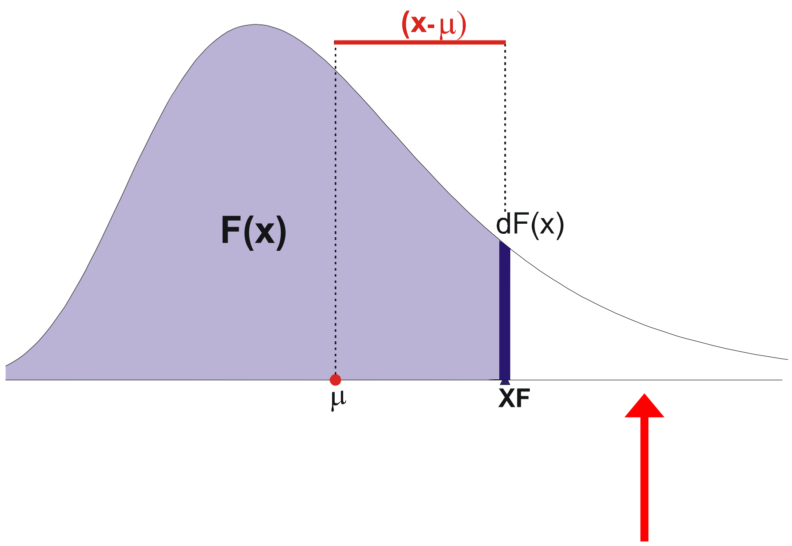
Simulation approach used for the second *L*-moment derivation of the inverse Gaussian distribution

Introduction

Flood Frequency Analysis (FFA) = estimation of upper quantiles of peak flows probability distribution, obtained from annual or partial duration series; assumed pdf is the statistical hypothesis



'Magnitude – return period' relationship

$$F\left(x_F(T)\right) = P\left(X \le x_F(T)\right) = 1 - 1/T$$

T - return period

$x_F(T)$ - flood magnitude of return period T years

$T = 10 \leftrightarrow F = 0.9$

$T = 100 \leftrightarrow F = 0.99$

$T = 1000 \leftrightarrow F = 0.999$

$$F\left(x_F\right) = \int_{-\infty}^{x_F} f\left(t\right) dt\,,\ f \text{ - probablity density function}$$

$$x_F = \inf\{x \in X : F(x) \ge F\} \equiv F^{\text{th}} \text{quantile}$$

Introduction

Frequency distribution function in FFA:

- **without upper bound**
- **with non-negative skewness**

For at-site FFA two-parameter distributions are recommended (e.g. Cunnane, 1989)

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Skewness is expressed by dimensionless dispersion measure

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Accuracy of dispersion measure is particularly important

Second *L*-moment vs standard deviation

DISPERSION MEASURES about the mean

Dispersion measure	Population	Sample
Standard deviation	$\sigma = \left[\int_{-\infty}^{+\infty} (x-\mu)^2 \, dF(x) \right]^{1/2}$	$s = \left[\frac{1}{N} \sum_{i=1}^{N} \left(x_i - \bar{x}\right)^2 \right]^{1/2}$
Second *L*-moment	$\lambda_2 = \int_{-\infty}^{+\infty} 2(x-\mu)F(x)\, dF(x)$	$l_2 = \frac{1}{N} \sum_{i=1}^{N} \frac{(2i-N-1)}{(N-1)} x_{i:N}$

DIMENSIONLESS DISPERSION MEASURES = VARIATION COEFFICIENTS

$$C_V = \sigma / \mu \overset{2-\text{param}}{=} g(\text{shape})$$

$$LC_V = \lambda_2 / \lambda_1 \overset{2-\text{param}}{=} h(\text{shape})$$

$$\lambda_1 \equiv \mu$$

Second *L*-moment vs standard deviation

ADVANTAGES OF SECOND *L*-MOMENT:

- **small bias of estimators**
- **no algebraic bound of estimator ratios, e.g. of *L*$\hat{C}$v**
- **existence when the mean of population exists**
- **robustness to the largest sample elements**

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- **SECOND *L*-MOMENT IS HIGHLY COMPETITIVE**
- ***L*-MOMENTS SYSTEM IS RECOMMENDED TO USE IN FFA (e.g. Robson & Reed, 1999)**

↓ **GOAL**

FIND SECOND *L*-MOMENT (*L*C$_v$) FOR „flood-like" PDF's which don't have derived *L*-moments

Inverse Gaussian

Inverse Gaussian = Convective Diffusion = Halphen A

$$f(x) = \frac{\alpha}{\sqrt{\pi x^3}} \exp\left[-\left(\alpha - \frac{\beta}{\alpha} x \right)^2 / x \right]$$

$\alpha, \beta > 0; \quad x > 0$

Cdf $F(x)$ & quantile x_F have no explicit analytical form

No L-moments have been derived

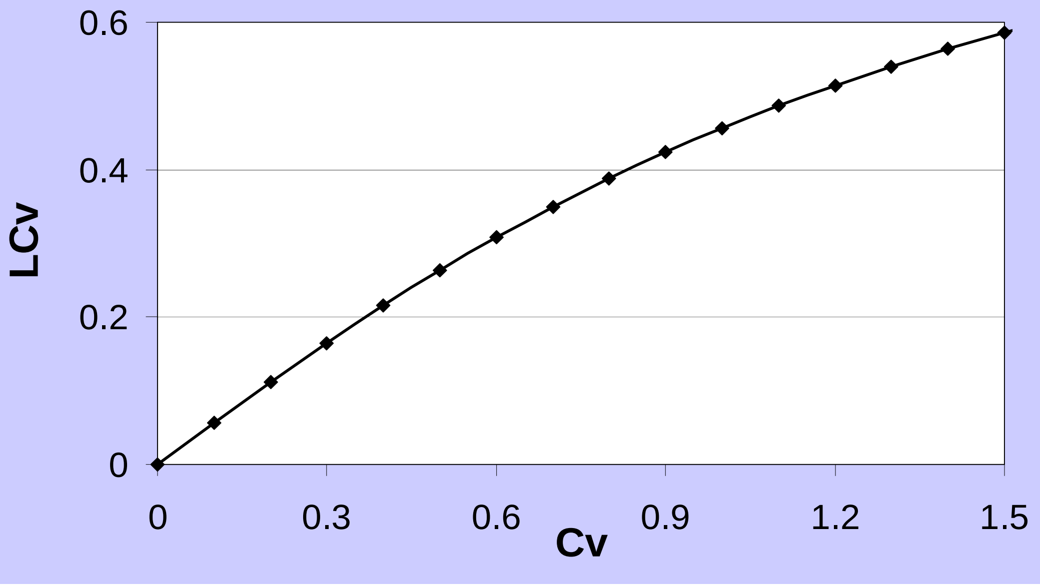
Pdf	Mean	Standard deviation	Variation coefficient
Inverse Gaussian	$\mu = \alpha^2 / \beta$	$\sigma = \alpha^2 \left(2\beta^3 \right)^{-1/2}$	$C_V = (2\beta)^{-1/2}$

Second *L*-moment derivation

Interest: $LC_V = h(\text{shape} = \beta), h = ??$

Methodology:

$$\left.\begin{array}{l} \mu = 1.0 \\ C_V = 0.1(0.1)1.5 \Rightarrow \beta \\ N = 50000 \end{array}\right\} \Rightarrow L\hat{C}_V \approx LC_V \Rightarrow \left(C_{V\,i}, LC_{V\,i}\right)\!\left(\beta_i\right), i = 1,\ldots,16$$



Second *L*-moment derivation

Polynomial least square fitting procedure (e.g. Krysicki et al., 1999)

$$LC_V = -0.1258 \cdot (C_V)^2 + 0.5823 \cdot C_V$$
$$R = 0.9998$$

$$LC_V = 0.0153 \cdot (C_V)^3 - 0.1654 \cdot (C_V)^2 + 0.6054 \cdot C_V$$
$$R = 0.9999$$

$$LC_V = 0.0157 \cdot (C_V)^4 - 0.0419 \cdot (C_V)^3 - 0.102 \cdot (C_V)^2 + 0.5849 \cdot C_V$$
$$R = 1.0$$

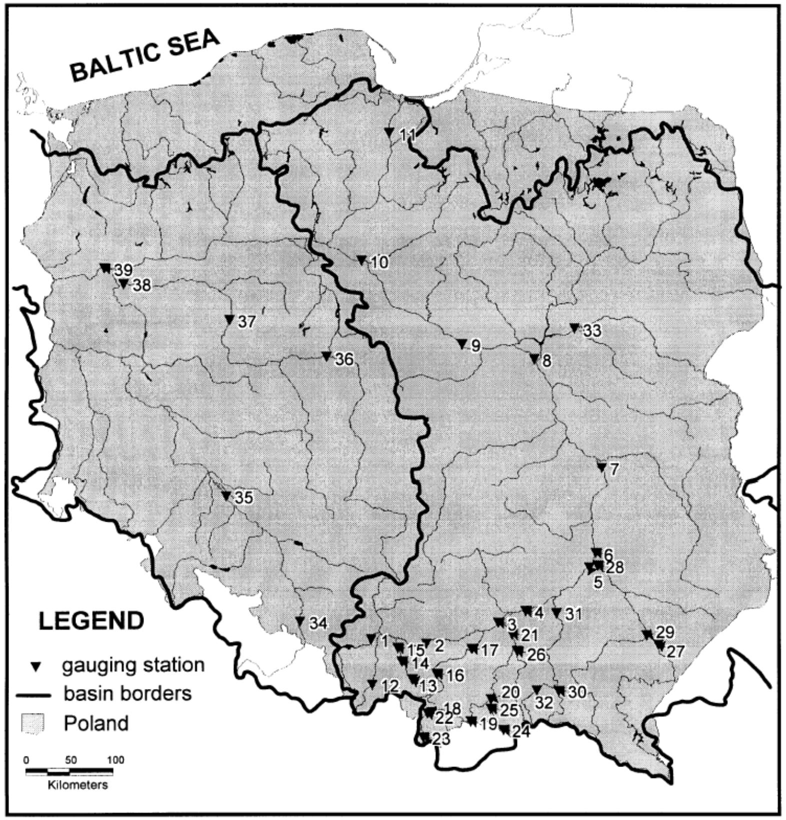
Pdf	Mean	Standard deviation	Variation coefficient
Inverse Gaussian	$\mu = \alpha^2 / \beta$	$\sigma = \alpha^2 \left(2\beta^3\right)^{-1/2}$	$C_V = \left(2\beta\right)^{-1/2}$

Range confirmation

$C_V \in (0, 1.5) \sim LC_V \in (0, 0.6)$

For 36 series of annual peak flows of Polish rivers, at least 85 years long measurements:

$$\mu\left(\hat{C}_V\right) = 0.634, \sigma\left(\hat{C}_V\right) = 0.135$$
$$\mu\left(L\hat{C}_V\right) = 0.328, \sigma\left(L\hat{C}_V\right) = 0.065$$



Methodology confirmation

N=10000	gamma			log-Gumbel		
C_V	$\hat{C}_V$	LC_V	$L\hat{C}_V$	$\hat{C}_V$	LC_V	$L\hat{C}_V$
0.2	0.200	0.112	0.112	0.201	0.100	0.100
0.6	0.600	0.324	0.325	0.595	0.241	0.240
1.0	1.008	0.500	0.502	0.924	0.315	0.320
2.0	2.010	0.763	0.761	1.339	0.380	0.380

Conclusions

- ✓ **Linear moments system has several desirable properties in respect to FFA (e.g. very small bias of sampling *L*-moments).**
- ✓ **Lack of analytical form of *L*-moments for some distributions precludes from using *L*-moments method for them.**
- ✓ **In respect to small bias of *L*-moments estimates, the simulation experiment can be used to derive the population *L*-moments.**
- ✓ **For inverse Gaussian second degree polynomial gives sufficiently good approximation of** $LC_V = h(C_V) \Rightarrow LC_V = h'(\beta)$ **and vice versa.**
- ✓ **Comparison of** LC_V **obtained from simulation approach and from analytical formula for distributions having the explicit *L*-moments, confirms the validity and high accuracy of the methodology.**

Plans

- to apply the methodology for *L*-moments (of any order) derivation for two- and three-parameter distributions having no analytical form of *L*-moments
- to elaborate the estimation method of large quantiles based on *L*-moments

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- Cunnane, C., 1989, *Statistical distributions for flood frequency analysis,* Operational Hydrology Report No.33, World Meteorological Organization, Geneva.
- Robson, A. and D., Reed, 1999, *Flood Estimation Handbook. Volume 3: Statistical procedures for flood frequency estimation,* Institute of Hydrology, Wallingford, Oxfordshire, UK.
- Krysicki W., J. Bartos, W. Dyczka, K. Królikowska, and M. Wasilewski, 1999, *Rachunek prawdopodobieństwa i statystyka matematyczna w zadaniach. II. Statystyka matematyczna*, 4th ed., PWN, Warsaw.

THANK YOU