INSTITUTO SUPERIOR TÉCNICO

2D Simulation of Discontinuous Shallow Flows

Hyperconcentrated Flows and Non-Newtonian Rheologies

- Conceptual Model
  - Geomorphic Flows
  - Conservation Equations
  - Closure models
- Discretization Scheme
  - General Scheme
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Geomorphic Flows

Geomorphic flows are governed by the volume and properties of the fluid matrix comprised of fluid and granular particles. These properties depend greatly on sediment concentration, size fraction and clay content.

Conceptual Model – The SWE system

The depth averaged system is used to model the flow:

$$\partial_t(h)+\partial_x(hu)+\partial_y(hv)=-\partial_t(Z_b)$$

$$\partial_t(uh)+\partial_x\left(u^2h+\frac{1}{2}gh^2\right)+\partial_y(uvh)=\frac{1}{\rho}\left(\partial_x(T_{11})-\partial_y(T_{12})-\rho g\partial_x(Z_b)-\tau_{b,x}\right)$$

$$\partial_t(vh)+\partial_y\left(v^2h-\frac{1}{2}gh^2\right)+\partial_x(\rho uvh)=\frac{1}{\rho}\left(\partial_y(T_{32})-\partial_x(T_{31})-\rho g\partial_y(Z_b)-\tau_{b,y}\right)$$

$$(1-p)\partial_t(Z_b)=D-E$$

Where

$\rho=\rho^{(w)}(1+C(s-1))$

C – Sediment concentration

s – Sediment specific weight

D – Deposition

E – Erosion

u, v – x and y velocity components

Z_b – Bottom elevation

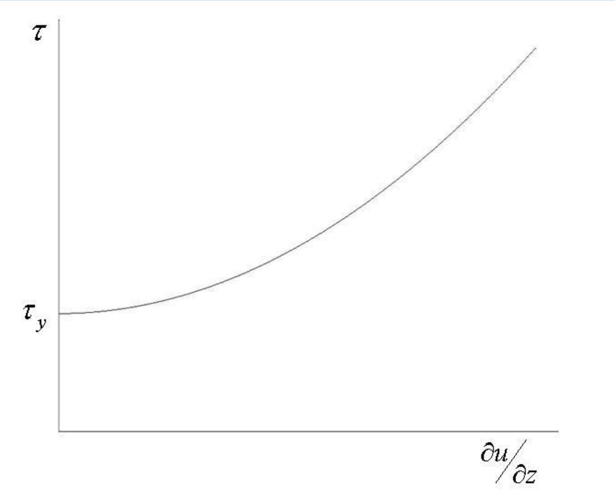
T_{ij} – Depth integrated turbulent stresses

Closure Models – Rheology Reconstruction

Rheology compatible with the simplification of a single-phase, continuum description of the flow are introduced with the bottom friction source terms

$$\tau_b = \tau_y + \tau_v + \tau_t + \tau_d$$

$$\tau_b = \tau_y + \mu \frac{\partial u}{\partial z} + C_f \left(\frac{\partial u}{\partial z} \right)^2$$



Shear-thickening fluid with Bigham fluid stress threshold

Closure Models – Turbulence and Mass fluxes

Turbulent stresses will be approximated with a simplified version of the k-ε model

The bottom interaction will be constructed around the closure equations derived in previous work by Ferreira (2005)

Discretization Scheme

The set of requirements for a two dimensional flow river simulation can be enumerated as

- Able to handle complex topography;
- Dry bed advancing fronts;
- Wetting and drying moving boundaries;
- High roughness values;
- Steady or unsteady flow;
- Subcritical or supercritical conditions.

As a consequence, robust schemes are fundamental to achieve satisfactory solutions quality wise, but the computational time is also a concern.

Discretization Scheme

The 2DH SWE equation system in compact notation can be written as

$$\partial_t\left(\mathbf{U}(\mathbf{V})\right)+\nabla.\mathbf{E}(\mathbf{U})=\mathbf{H}(\mathbf{U})\Leftrightarrow\partial_t\left(\mathbf{U}(\mathbf{V})\right)+\partial_x\left(\mathbf{F}(\mathbf{U})\right)+\partial_y\left(\mathbf{G}(\mathbf{U})\right)=\mathbf{H}(\mathbf{U})$$

$$\partial_t\int_{\Omega_i}\mathbf{U}(\mathbf{V})dS+\int_{\Omega_i}\nabla\cdot\mathbf{E}(\mathbf{U})dS=\int_{\Omega_i}\mathbf{H}(\mathbf{U})dS$$

$$\partial_t\int_{\Omega_i}\mathbf{U}(\mathbf{V})dS+\oint_{L_i}\mathbf{E}\left(\mathbf{U}\right)\cdot ndl=\int_{\Omega_i}\mathbf{H}(\mathbf{U})dS$$

$$\partial_t A_i\left\langle\mathbf{U}_i\right\rangle+\oint_{L_i}\mathbf{E}\left(\mathbf{U}\right)\cdot ndl=A_i\left\langle\mathbf{H}_i\right\rangle$$

$$\partial_t A_i\left\langle\mathbf{U}_i\right\rangle+\sum_{k=1}^{n_i}L_k\left\langle\mathbf{E}\cdot n\right\rangle_{ik}=A_i\left\langle\mathbf{H}_i\right\rangle$$

Where

$$\left\langle\mathbf{U}_i\right\rangle=\frac{1}{A_i}\int_{\Omega_i}\mathbf{U}\left(x,y,t\right)dS$$

$$\mathbf{E}\cdot n=\mathbf{F}n_x+\mathbf{G}n_y$$

Discretization Scheme

The fluxes trough the *k* edge of cell *i* represent the differences between the values of the independent variables on the adjacent cells *j* and *i*, separated by edge *k*

$$\langle \mathbf{F} \rangle_{ik} = \Delta_{ik} \langle \mathbf{F} \rangle = \langle \mathbf{F}_j \rangle - \langle \mathbf{F}_i \rangle$$

$$\langle \mathbf{G} \rangle_{ik} = \Delta_{ik} \langle \mathbf{G} \rangle = \langle \mathbf{G}_j \rangle - \langle \mathbf{G}_i \rangle$$

$$\langle \mathbf{E}\cdot n \rangle_{ik} = \Delta_{ik} \langle \mathbf{E}\cdot n \rangle = \left(\langle \mathbf{E}_j \rangle - \langle \mathbf{E}_i \rangle \right)\cdot n_{ik}$$

The local discretization of the system becomes

$$A_i \frac{\Delta \langle \mathbf{U}_i \rangle}{\Delta t} + \sum_{k=1}^{n_i} L_k \Delta_{ik} \langle \mathbf{E}\cdot n \rangle = A_i \langle \mathbf{H}_i \rangle$$

Discretization Scheme

The flux variations can be expressed as a function of the independent conservative variables using a Jacobian matrix orthogonal to the edge in question, i.e.

$$\Delta_{ik} \left\langle \mathbf{E} \cdot n \right\rangle = \mathbf{J}_{n,ik} \Delta_{ik} \left\langle \mathbf{U} \right\rangle \text{ with } \mathbf{J}_{n,ik} = \frac{\partial \mathbf{E} \cdot n_{ik}}{\partial \mathbf{U}} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} n_x + \frac{\partial \mathbf{G}}{\partial \mathbf{U}} n_y$$

The flux vectors are not homogeneous functions of the dependant variables, for the shallow water type system, leading to an approximation based on a local linearization of the system

$$\Delta_{ik} \left\langle \mathbf{E} \cdot n \right\rangle = \breve{\mathbf{J}}_{n,ik} \, \Delta_{ik} \left\langle \mathbf{U} \right\rangle$$

Discretization Scheme

Following Roe [1981]

$$\breve{u}_{ik} = \frac{u_i\sqrt{h_i} + u_j\sqrt{h_j}}{\sqrt{h_i} + \sqrt{h_j}}; \quad \breve{v}_{ik} = \frac{v_i\sqrt{h_i} + v_j\sqrt{h_j}}{\sqrt{h_i} + \sqrt{h_j}}; \quad \breve{c}_{ik} = \sqrt{g\frac{h_i + h_j}{2}}$$

$$\breve{\lambda}_{ik}^{(1)} = \left(\breve{\mathbf{u}}\cdot n - \breve{c}\right)_{ik} \; ; \; \breve{\lambda}_{ik}^{(2)} = \left(\breve{\mathbf{u}}\cdot n\right)_{ik} \; ; \; \breve{\lambda}_{ik}^{(3)} = \left(\breve{\mathbf{u}}\cdot n + \breve{c}\right)_{ik}$$

$$\breve{\mathbf{e}}_{ik}^{(1)} = \begin{bmatrix} 1 \\ \breve{u} - \breve{c}\,n_x \\ \breve{v} - \breve{c}\,n_y \end{bmatrix}_{ik} \; ; \; \breve{\mathbf{e}}_{ik}^{(2)} = \begin{bmatrix} 0 \\ -\breve{c}\,n_y \\ \breve{c}\,n_x \end{bmatrix}_{ik} \; ; \; \breve{\mathbf{e}}_{ik}^{(3)} = \begin{bmatrix} 1 \\ \breve{u} + \breve{c}\,n_x \\ \breve{v} + \breve{c}\,n_y \end{bmatrix}_{ik}$$

Discretization Scheme

The independent variables variation is projected on a new base, formed by the eigenvectors of the system

$$\Delta_{ik}\left\langle \mathbf{U} \right\rangle = \sum_{n=1}^{3} \alpha_{ik}^{(n)}\, \breve{\mathbf{e}}_{ik}^{(n)}$$

The fluxes are also projected on the the homogeneous system eigenvectors base

$$\Delta_{ik}\left\langle \mathbf{E}\cdot n \right\rangle = \left(\left\langle \mathbf{E}_j \right\rangle - \left\langle \mathbf{E}_i \right\rangle\right)\cdot n_{ik} = \sum_{n=1}^{3} \breve{\lambda}_{ik}^{(n)}\, \alpha_{ik}^{(n)}\, \breve{\mathbf{e}}_{ik}^{(n)}$$

The wave strengths are

$$\alpha_{ik}^{(1)} = \frac{\Delta_{ik}\left\langle h \right\rangle}{2} - \frac{1}{2\breve{c}_{ik}}\left(\Delta_{ik}\left\langle \mathbf{u}h \right\rangle - \breve{\mathbf{u}}_{ik}\,\Delta_{ik}\left\langle h \right\rangle\right) n_{ik}$$

$$\alpha_{ik}^{(2)} = \frac{1}{\breve{c}_{ik}}\left(\Delta_{ik}\left\langle \mathbf{u}h \right\rangle - \breve{\mathbf{u}}_{ik}\,\Delta_{ik}\left\langle h \right\rangle\right) t_{ik}$$

$$\alpha_{ik}^{(3)} = \frac{\Delta_{ik}\left\langle h \right\rangle}{2} + \frac{1}{2\breve{c}_{ik}}\left(\Delta_{ik}\left\langle \mathbf{u}h \right\rangle - \breve{\mathbf{u}}_{ik}\,\Delta_{ik}\left\langle h \right\rangle\right) n_{ik}$$

Discretization Scheme

In the Roe approach, the solution is obtained by exactly solving a series of Riemann Problems derived from the local linearization of the initial hyperbolic, non-homogeneous and non-linear problem at every edge with two corresponding cells.

In the 2D space, the solution is obtained by reducing each RP at each edge to a 1D RP projected in the orthogonal direction to the edge.

To Account for the source terms

$$A_i \frac{\Delta \langle \mathbf{U}_i \rangle}{\Delta t} + \sum_{k=1}^{n_i} L_k \Delta_{ik} \langle \mathbf{E} \cdot n \rangle = A_i \langle \mathbf{H}_i \rangle \Rightarrow A_i \frac{\Delta \mathbf{U}_i}{\Delta t} + \sum_{k=1}^{3} L_k (\Delta \mathbf{E} - \mathbf{T})_{ik} \cdot n_{ik} = 0$$

Discretization Scheme

The numerical fluxes from the bottom variation, ***T***,are also discretized in the same conservative way as the advective flux terms

$$A_i \mathbf{H}_i = \sum_{k=1}^{3} \sum_{n=1}^{3} \beta_{ik}^{(n)} \breve{\mathbf{e}}_{ik}^{(n)} L_k$$

Leading to the final form of, a now numerically homogenous, system

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^{n} - \frac{\Delta t}{A_i} \sum_{k=1}^{3} L_k \sum_{n=1}^{3} \left(\breve{\lambda}^{(n)} \alpha^{(n)} - \beta^{(n)} \right)_{ik}^{-} \breve{\mathbf{e}}_{ik}^{(n)}$$

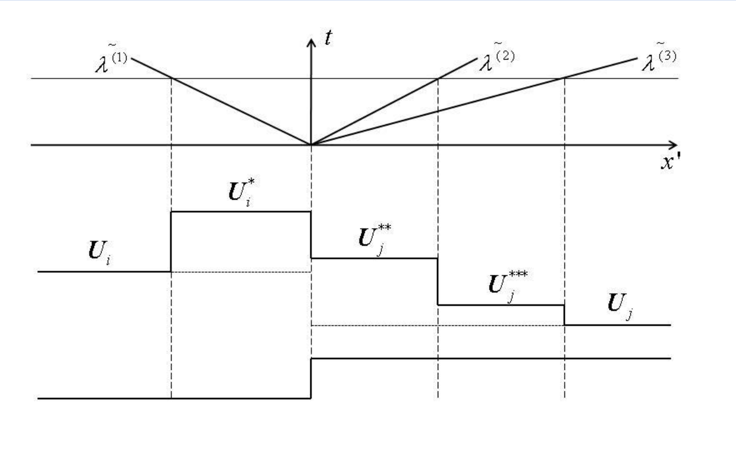
Time Step – Stability Region

A classical Courant-Friedrichs-Lewy (CFL) condition is used to deal with the advective part of the flux

$$\chi_i = \frac{A_i}{\max\left(L_k\right)_{i,k=1,2,3}} \qquad \Delta t \leq CFL\, \Delta t^{\tilde{\lambda}} \qquad \Delta t^{\tilde{\lambda}} = \frac{\min\left(\chi_i, \chi_j\right)}{\max|\tilde{\lambda}^{(n)}|_{n=1,2,3}}$$

Influence of the bottom source terms in the stability region?

Murillo 2010 described new intermediate states generated in the solution of the RP



Time Step – Stability Region

In order to force stability and the independence of each RP

$$\Delta t^{***} = \frac{\chi_j}{2\,\breve{\lambda}_{ik}^{(3)}} \frac{h_j^n}{h_j^n - h_j^{***}} \quad \text{if} \quad h_j^{***} < 0 \;\text{ and }\; h_j^n \neq 0$$

$$\Delta t^{*} = \frac{\chi_i}{2\,\breve{\lambda}_{ik}^{(1)}} \frac{h_i^n}{h_i^n - h_i^{*}} \quad \text{if} \quad h_i^{*} < 0 \;\text{ and }\; h_i^n \neq 0$$

$$\Delta t \leq \begin{cases} min\left(\Delta t^{***}, \Delta t^{*}, \Delta t^{\breve{\lambda}}\right) & \text{if} \left(\breve{\lambda}^{(1)} \; \breve{\lambda}^{(3)}\right)_{ik} \leq 0 \\ \Delta t^{\breve{\lambda}} & \text{otherwise} \end{cases}$$

Treatment of the Source Terms

Bottom source terms

$$\beta_{ik}^{(1)} = -\frac{1}{2\tilde{c}_{ik}}\left(\frac{p_b}{\rho_w}\right); \quad \beta_{ik}^{(2)} = 0; \quad \beta_{ik}^{(3)} = -\beta_{ik}^{(1)}$$

$$\left(\frac{p_b}{\rho_w}\right)_k^a = -g\left(\breve{h}\,\delta z\right)_{ik} \qquad \left(\frac{p_b}{\rho_w}\right)_k^b = -g\left(h_r - \frac{|\delta z_{ik}|}{2}\right)\delta z_{ik'}$$

Optimization of the solution, forcing energy dissipating solution when necessary.

To ensure positivity of the solution, the numerical flux can be locally reconstructed

$$h_i^* = h_i + \left(\alpha - \frac{\beta}{\breve{\lambda}}\right)_{ik}^{(1)} \geq 0$$

$$h_j^{***} = h_j - \left(\alpha - \frac{\beta}{\breve{\lambda}}\right)_{ik}^{(3)} \geq 0$$

$$\Rightarrow \quad \beta_{ik}^{(1)} \geq \beta_{min}^{(1)}, \quad \beta_{min}^{(1)} = -\left(h_i^n + \alpha_{ik}^{(1)}\right)|\,\breve{\lambda}_{ik}^{(1)}\,|$$

$$\beta_{ik}^{(3)} \geq \beta_{min}^{(3)}, \quad \beta_{min}^{(3)} = -\left(h_j^n - \alpha_{ik}^{(3)}\right)\breve{\lambda}_{ik}^{(1)}$$

Treatment of the Source Terms

Friction source terms

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{A_i} \sum_{k=1}^{3} L_k \sum_{n=1}^{3} \left(\tilde{\lambda}^{(n)} \alpha^{(n)} - \beta^{(n)} \right)_{ik}^{-} \tilde{\mathbf{e}}_{ik}^{(n)} + \Delta t \left(\mathbf{R}_i^{n+1} \right)$$

Where

$$\mathbf{R} = \begin{bmatrix} 0 & -\frac{\tau_{b,x}}{\rho} & -\frac{\tau_{b,y}}{\rho} \end{bmatrix}^T$$

$$\tau_b = \tau_y + \mu \frac{\partial u}{\partial z} + C_f \left(\frac{\partial u}{\partial z} \right)^2 \Rightarrow \begin{cases} \left(\tau_{b,x} \right)_i = \tau_y + \mu \frac{u_i}{h_i} + C_f \frac{u_i^2}{h_i^2} \\ \left(\tau_{b,y} \right)_i = \tau_y + \mu \frac{v_i}{h_i} + C_f \frac{v_i^2}{h_i^2} \end{cases}$$

Wetting and Drying Algorithm

It is still possible to find regions of the solution where negative values of water depth is expected, namely in wet/dry interfaces with discontinuous bed level.

To ensure positivity and conservation in the solution for all cases, the fluxes for the update of the conserved variables are computed as

$$\text{If } h_j^n = 0 \text{ and } h_j^{***} < 0 \quad \text{then } (\Delta\mathbf{E} - \mathbf{T})_{ik}^- = (\Delta\mathbf{E} - \mathbf{T})_k \text{ and } (\Delta\mathbf{E} - \mathbf{T})_{jk}^- = 0$$

$$\text{If } h_i^n = 0 \text{ and } h_i^{*} < 0 \quad \text{then } (\Delta\mathbf{E} - \mathbf{T})_{jk}^- = (\Delta\mathbf{E} - \mathbf{T})_k \text{ and } (\Delta\mathbf{E} - \mathbf{T})_{ik}^- = 0$$

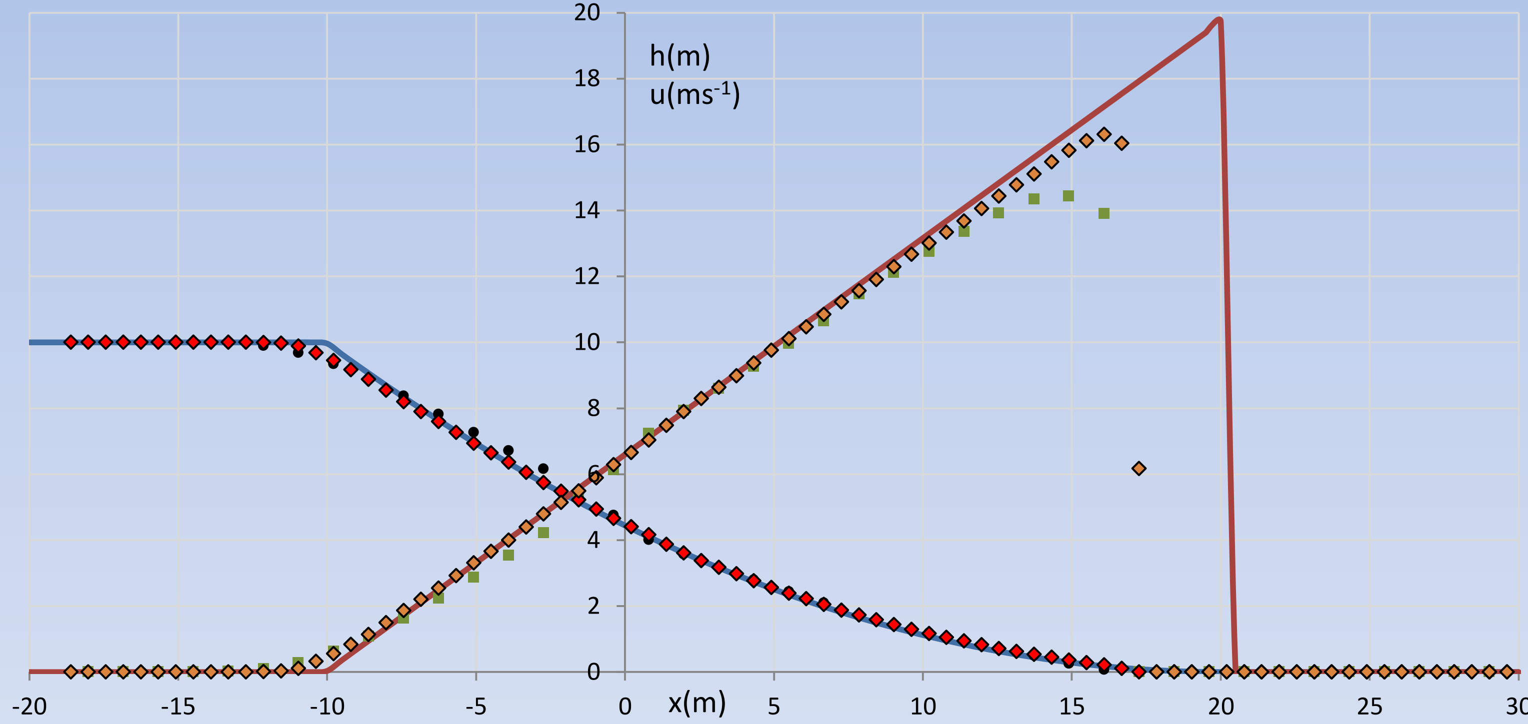
Results

Ritter Solution

$$h_L = 10 \ m$$

$$h_R = 0.0 \ m \qquad t = 1.0 \ s$$

$$CFL = 0.8$$



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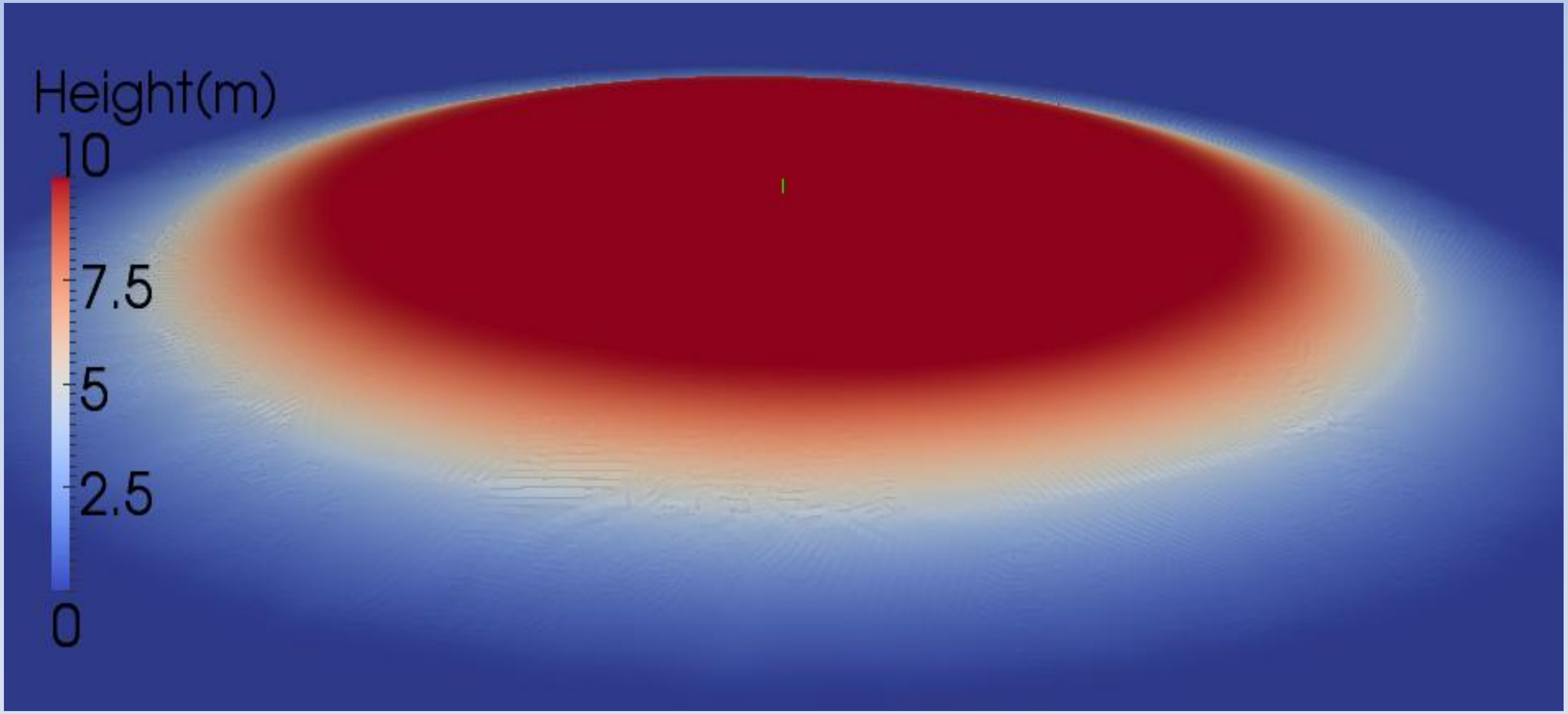

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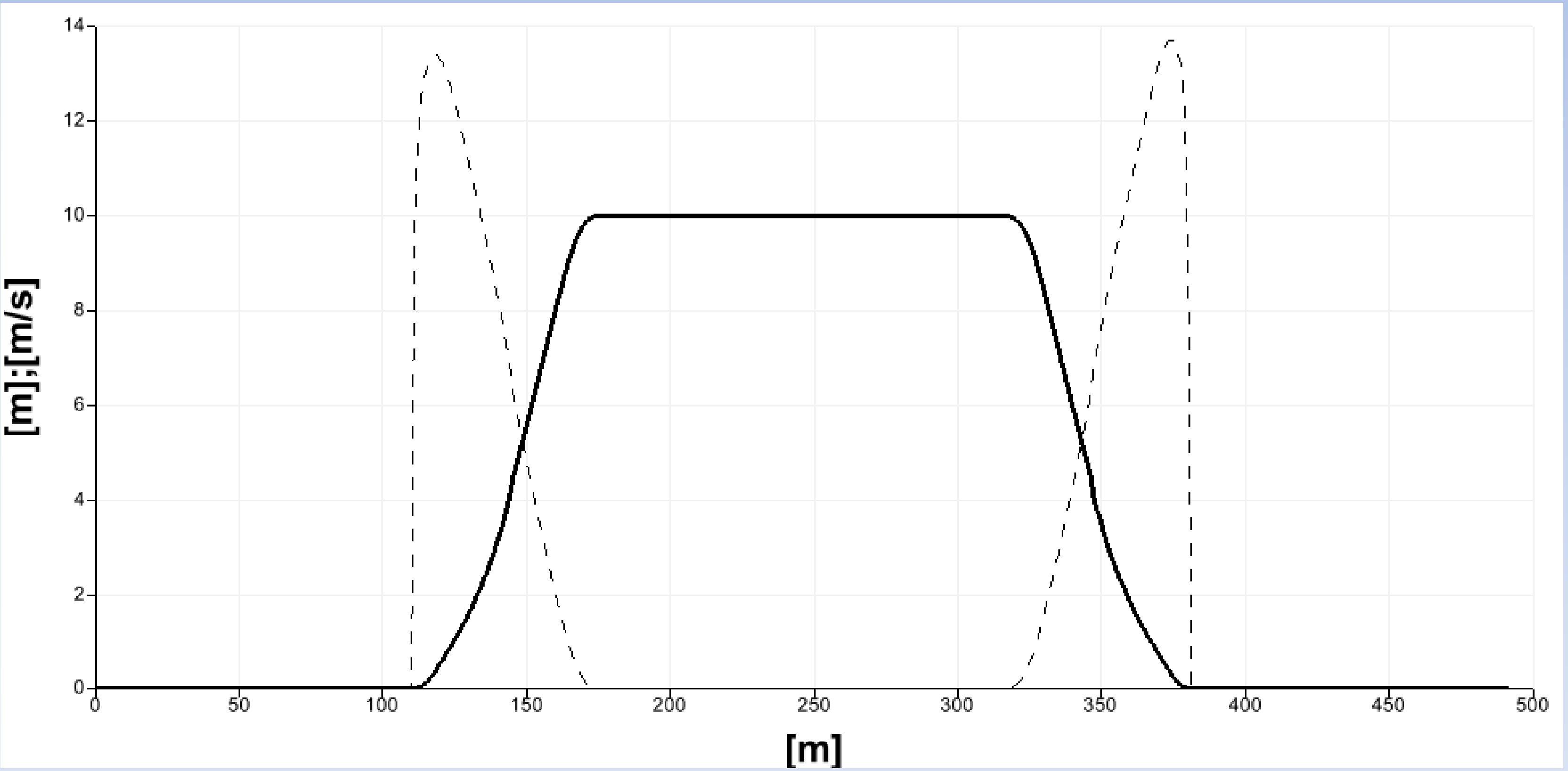


Results

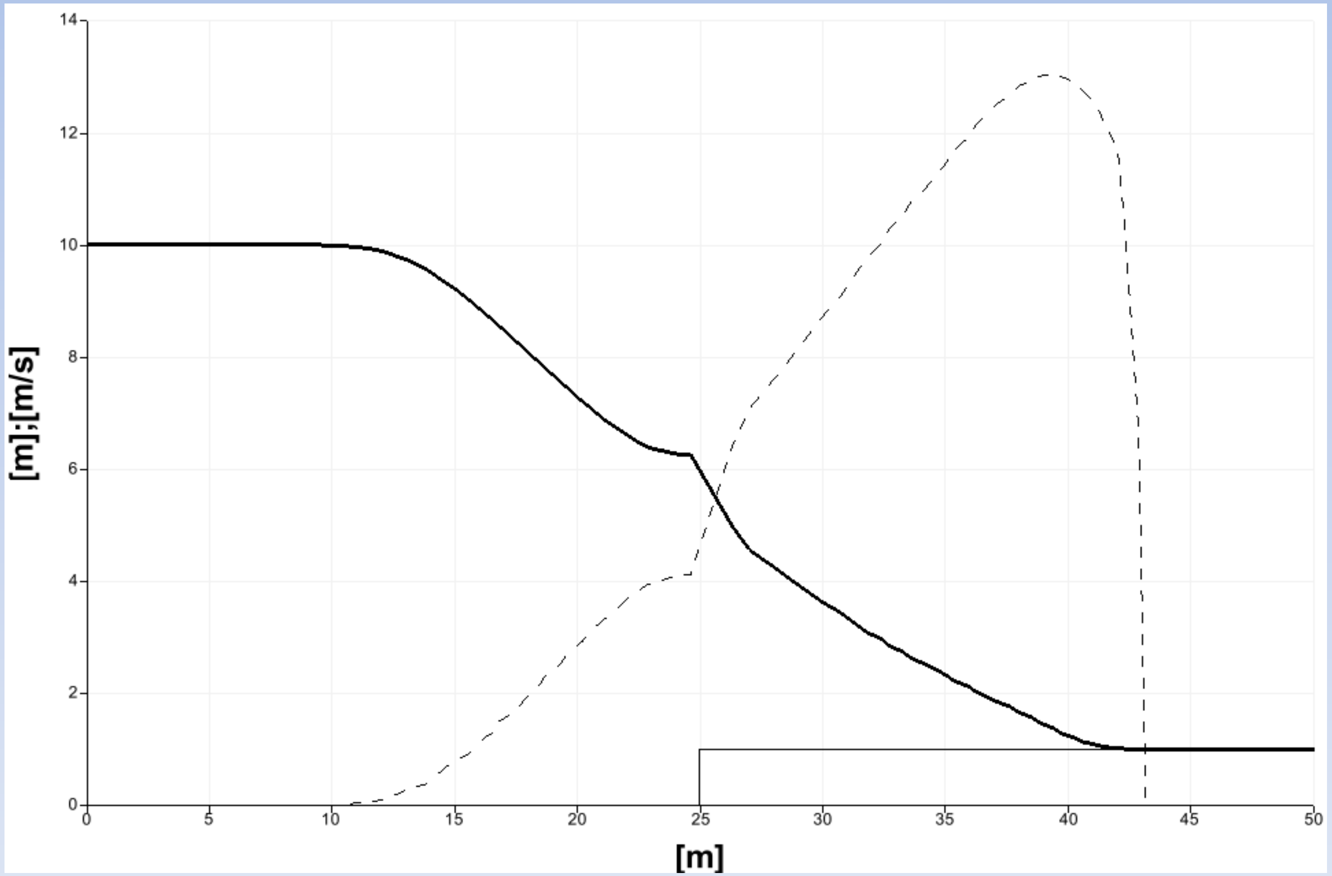
$h_L = 10\ m$

$h_R = 0.0\ m \qquad t = 1.0\ s$

$Z_{bR} = 1.0\ m$

$CFL = 0.8$

Discontinuous bottom dry bed solution



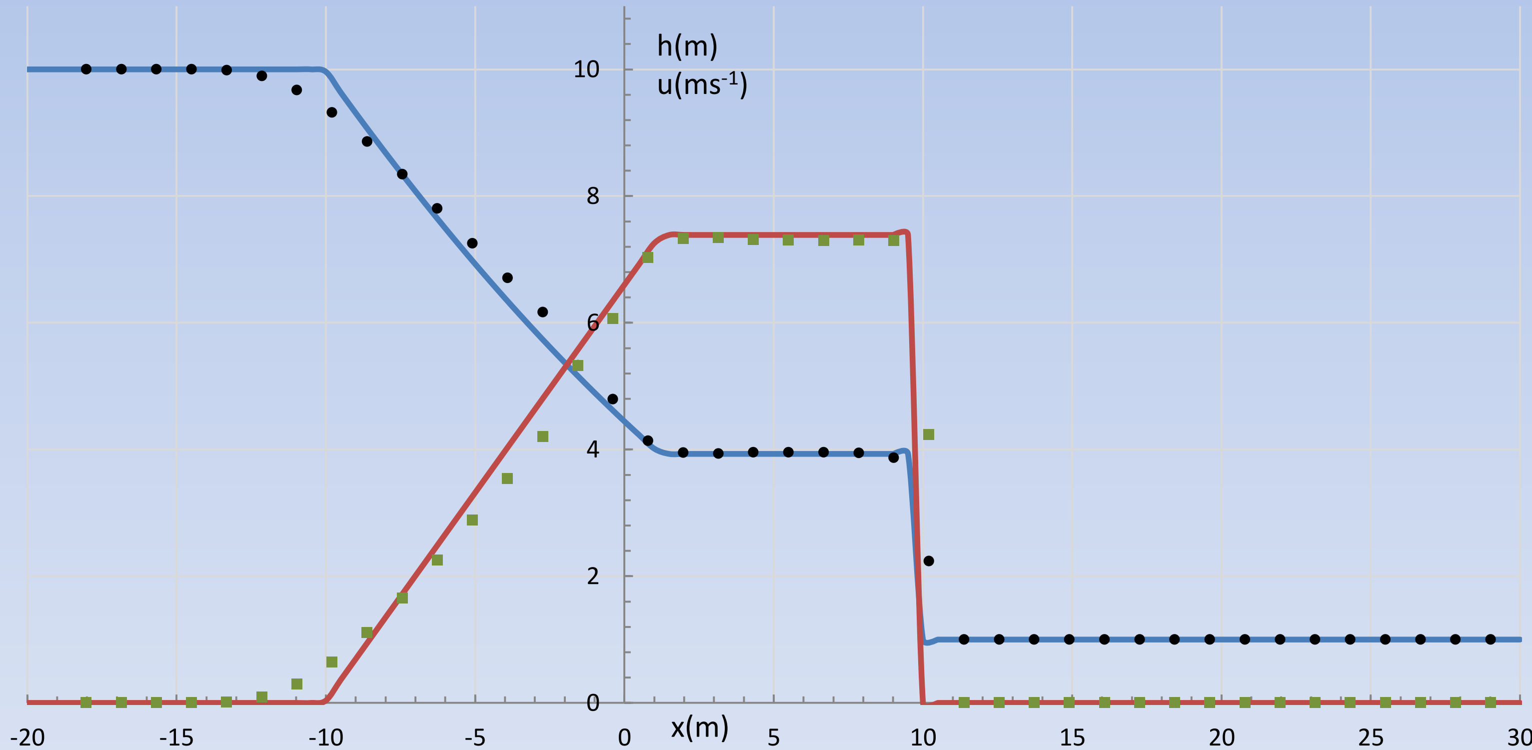
Results

Stoker Solution

$$h_L = 10 \ m$$

$$h_R = 1.0 \ m \qquad t = 1.0 \ s$$

$$CFL = 0.8$$



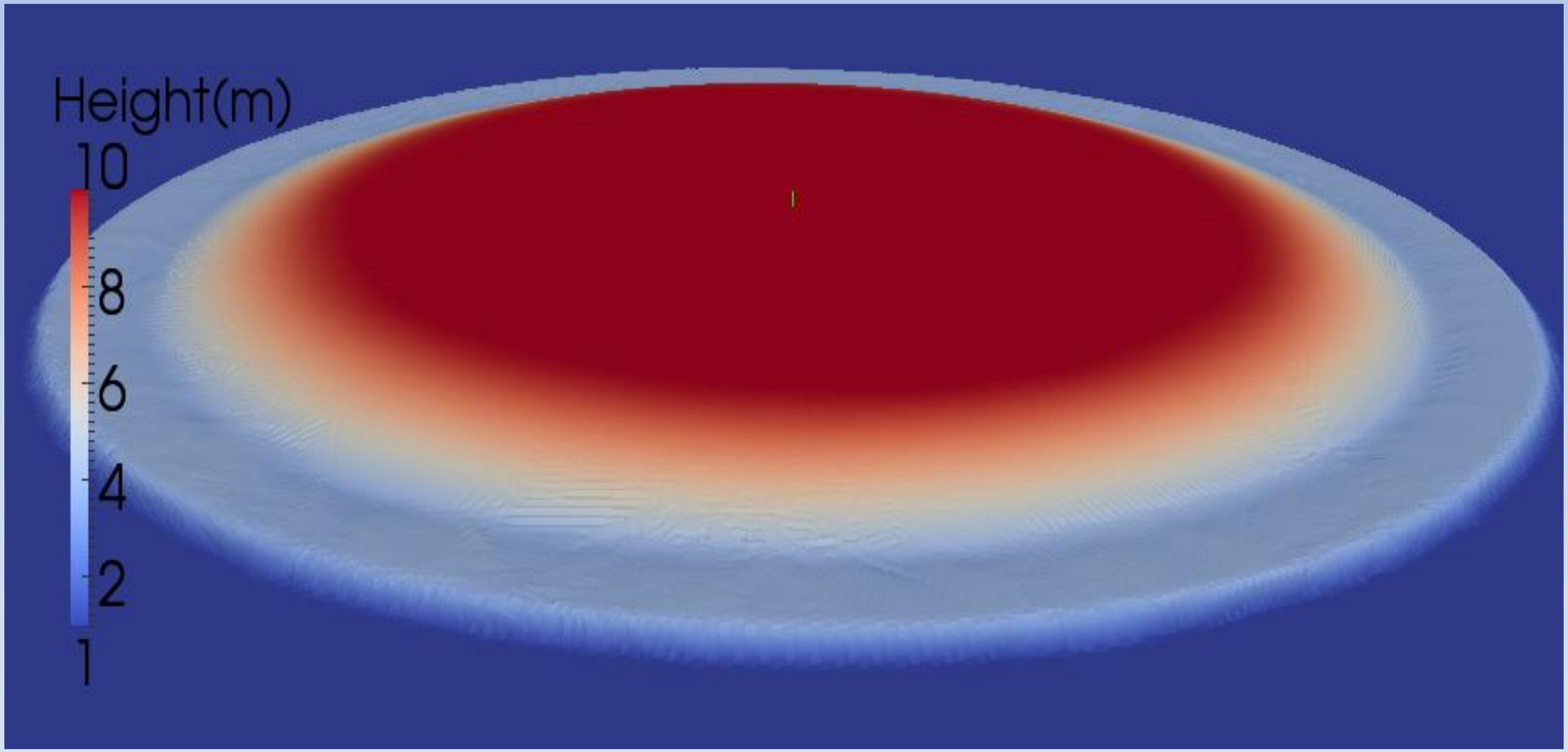
Results

Stoker Solution

$h_L = 10 \ m$

$h_R = 1.0 \ m \qquad t = 1.0 \ s$

$CFL = 0.8$



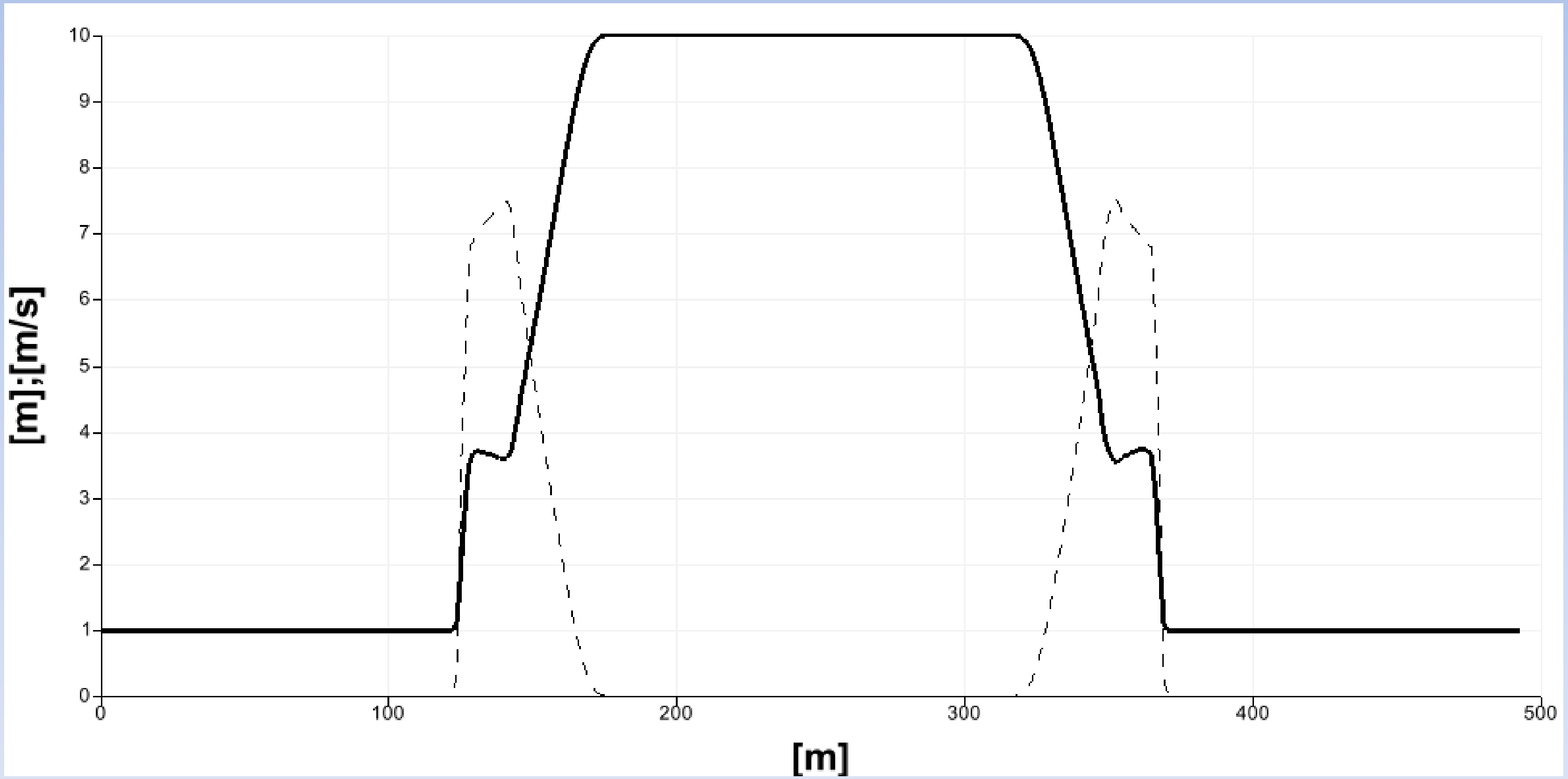
Results

Stoker Solution

$$h_L = 10 \ m$$

$$h_R = 1.0 \ m \qquad t = 1.0 \ s$$

$$CFL = 0.8$$



Conclusions

-An approximate Riemann solver, adapted to the structure of a system describing a highly unsteady discontinuous flow over complex geometries has been presented, were a new wave associated to the source terms is added;

-A reformulation of the stability condition that generalizes the classical CFL condition by including the influence of the source terms was presented;

-Flows presenting simple quasi-newtonian rheologies, like mud-flows are within the modeling capabilities of the scheme.

Further Developments

-The development and implementation of an uncoupled mobile bed solver, were a weak discontinuity associated with the bedload will be introduced in the solution;

-The discretization of the turbulent stresses, in order to model turbulent mudflows.