Numerical verification of Log-Law in flows with pressure gradient

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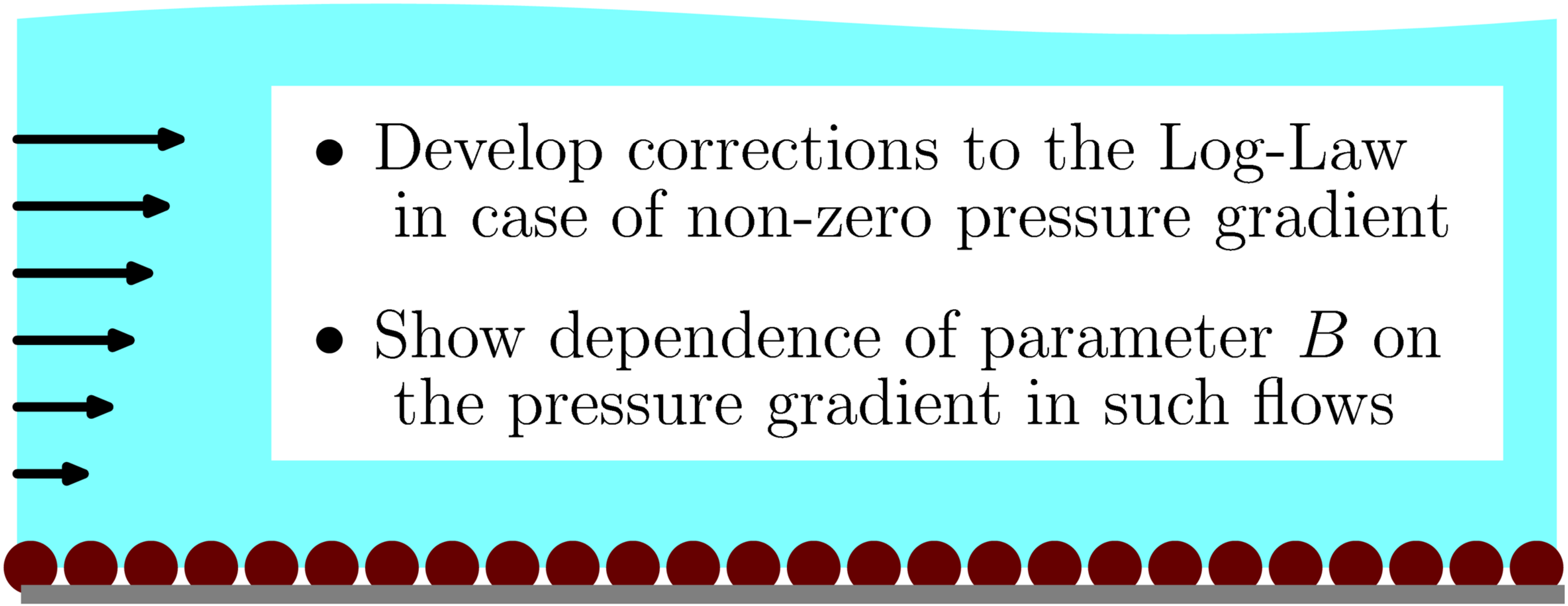
Mean velocity profile
Usage of Log-Law

$$\frac{U(y)}{U_*} = \frac{1}{\kappa} \ln \frac{y}{k} + B,$$

- **Nikuradze — sand roughness,**
- **Inaccurate if roughness is irregular (Yalin),**
- **Log-Law is standard approach to describe mean velocity profile**
- **2 layers — inner (at bed) and outer (upper core flow)**

Aims of the work

For non-uniform flows over rough bed :



- Develop corrections to the Log-Law in case of non-zero pressure gradient
- Show dependence of parameter B on the pressure gradient in such flows

Fundamental Physics

momentum equations :

$$U_j \frac{\partial U_j}{\partial x_i} + \frac{1}{\rho}\frac{\partial P}{\partial x_i} = F_i - \frac{\partial}{\partial x_j}\left(\overline{u_i u_j} - \nu \frac{\partial U_i}{\partial x_j}\right),\ i = 1, 2, 3$$

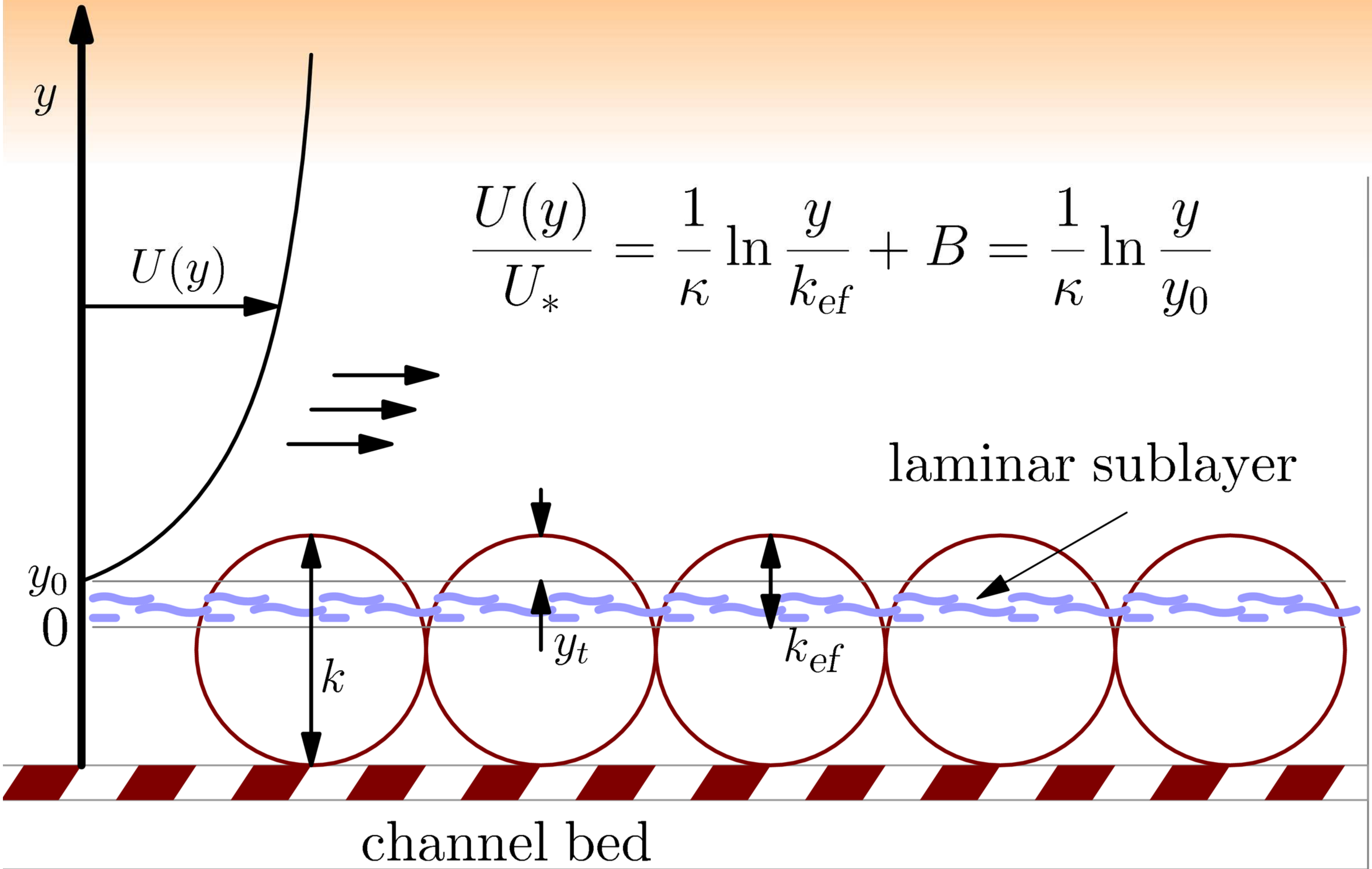
$$\frac{\partial}{\partial y}\left(-\overline{u'v'} + \nu \frac{\partial U}{\partial y}\right) = gS_0, \quad S_0 = \tan\theta$$

total shear stress τ:

$$\frac{\tau(y)}{\rho} \equiv -\overline{u'v'} + \nu \frac{\partial U}{\partial y} = ghS_0\left(1 - \frac{y}{h}\right),$$

Channel with spherical roughness

$$\frac{U(y)}{U_*} = \frac{1}{\kappa} \ln \frac{y}{k_{ef}} + B = \frac{1}{\kappa} \ln \frac{y}{y_0}$$



Log-Law in boundary layers

Prandtl assumed :

$$\rho \ell^2 \left(\frac{\partial U}{\partial y} \right)^2 = \tau(y) = \tau_{\text{const}}, \quad \ell = \kappa y, \quad \Rightarrow$$

$$\frac{\partial U}{\partial y} = \frac{\sqrt{\tau / \rho}}{\kappa y} \quad \Rightarrow \quad U(y) = \frac{U_*}{\kappa} \ln \frac{y}{y_0}$$

Log-Law in open Channel

$$\frac{\tau(y)}{\rho} = ghS_0\left(1 - \frac{y}{h}\right), \quad ghS_0 = U_*^2,$$

stress is *linear* in vertical

$$\kappa^2 y^2 \left(\frac{\partial U}{\partial y}\right)^2 = \frac{\tau(y)}{\rho} = U_*^2\left(1 - \frac{y}{h}\right),$$

$$\frac{\partial U}{\partial y} = \frac{U_*}{\kappa y}\sqrt{1 - \frac{y}{h}}$$

Log-Law in open Channel – 2

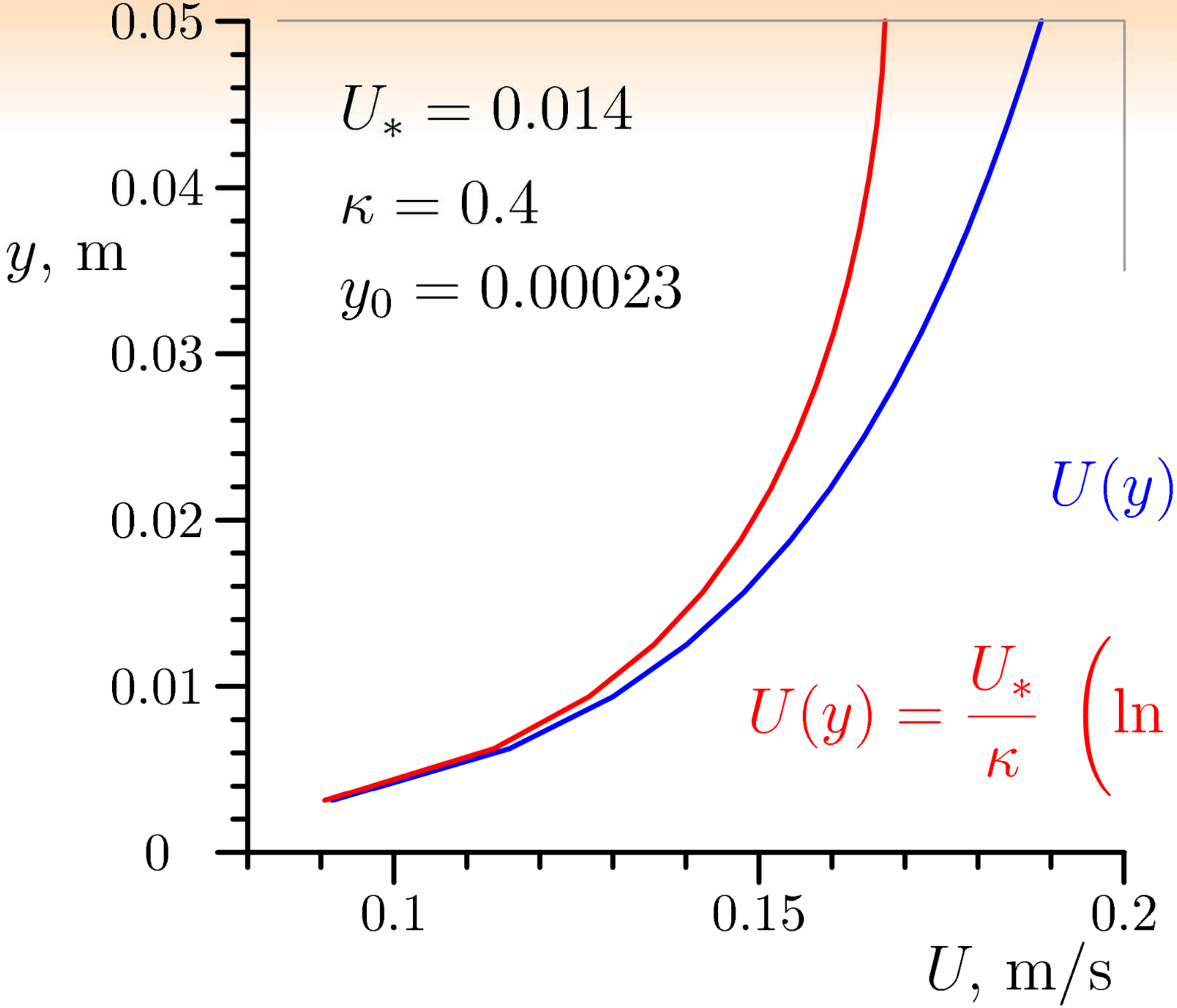
$$U(y) = \frac{U_*}{\kappa}\left[\ln\frac{y}{y_0} - 2\ \ln\frac{1+\sqrt{1-y/h}}{1+\sqrt{1-y_0/h}}\right.$$

$$\left.+2\left(\sqrt{1-y/h} - \sqrt{1-y_0/h}\right)\right] =$$

$$= \frac{U_*}{\kappa}\left(\ln\frac{y}{y_0} + A(y) - A(y_0)\right)$$

$$A(y) = 2\left[\sqrt{1-y/h} - \ln\left(1+\sqrt{1-y/h}\right)\right]$$

Log-Law in open Channel – 3



$$U(y) = \frac{U_*}{\kappa} \ln \frac{y}{y_0}$$

$$U(y) = \frac{U_*}{\kappa} \left(\ln \frac{y}{y_0} + A(y) - A(y_0) \right)$$

Flows in gravel-bed channel with pressure gradient

vertical velocity profile in

- $\frac{\partial P}{\partial x} = 0$ – **uniform zero pressure gradient flow**
- $\frac{\partial P}{\partial x} < 0$ – **accelerating flow**
- $\frac{\partial P}{\partial x} > 0$ – **decelerating flow**

Conditions for Non-ZPG flows

for non-uniform flows we assume

1) slope of channel bottom is small
2) pressure is hydrostatical in vertical
3) gradually varied flow, i.e. $U_*^2 = ghS_f$
4) flow is unidirectional with $(U + u, v)$
5) negligible x-derivatives of turbulent stresses

Shear Velocity, Non-ZPG flows

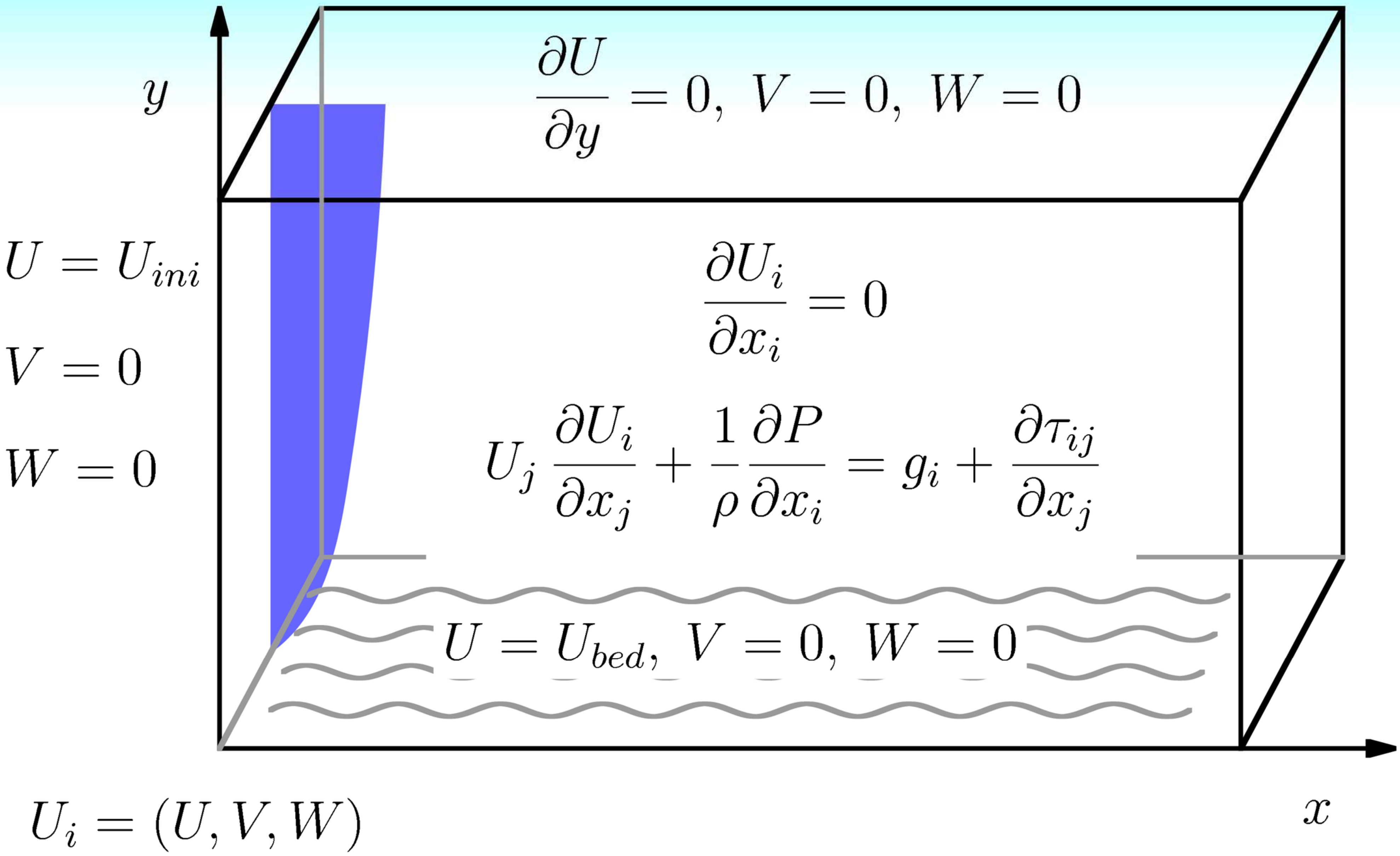
$$\begin{cases} U\dfrac{\partial U}{\partial x} + \dfrac{1}{\rho}\dfrac{\partial P}{\partial x} = gS_0 + \dfrac{\partial \tau/\rho}{\partial y}, \quad \tau/\rho = -\overline{u'v'} + \nu\dfrac{\partial U}{\partial y}, \\ P(y) = \rho g h \cos\theta \left(1 - \dfrac{y}{h}\right), \end{cases}$$

$$U\frac{\partial U}{\partial x} + \frac{g}{2}\frac{\partial h}{\partial x} = g(S_0 - S_f), \qquad gS_f = \tau_b/\rho h,$$

$$U\frac{\partial h}{\partial x} + h\frac{\partial U}{\partial x} = 0, \qquad \frac{\partial P}{\partial x} = \frac{1}{2}\rho g\frac{\partial h}{\partial x},$$

$$\boxed{U_*^2 = ghS_0 - \frac{\partial P}{\partial x}\frac{h}{\rho}\left(1 - 2\,\mathrm{Fr}^2\right)}$$

Numerical procedure



Experimental data

after M. Tehrani (1992)

case	rough. height, mm	depth, mm	mean velocity, mm/s	shear velocity, mm/s	effect. rough., mm	B_0	B_1
K1Q1	1.15	49.8	227.8	12.48	0.37	7.28	9.45
K1Q2	1.15	49.7	164.8	9.18	0.47	7.81	9.91
K1Q3	1.15	49.9	106.0	6.46	0.48	6.73	8.13
K6Q1	6	49.0	219.4	15.91	1.47	5.43	6.97
K6Q2	6	49.1	154.6	13.15	1.475	5.34	5.36
K6Q3	6	48.9	111.0	8.46	1.37	5.23	6.33
K12Q2	12	46.9	155.9	14.82	2.72	5.35	5.7
K12Q3	12	49.7	106.0	9.21	2.73	5.30	6.11

Seek B algorithm

Iterative series of steps :

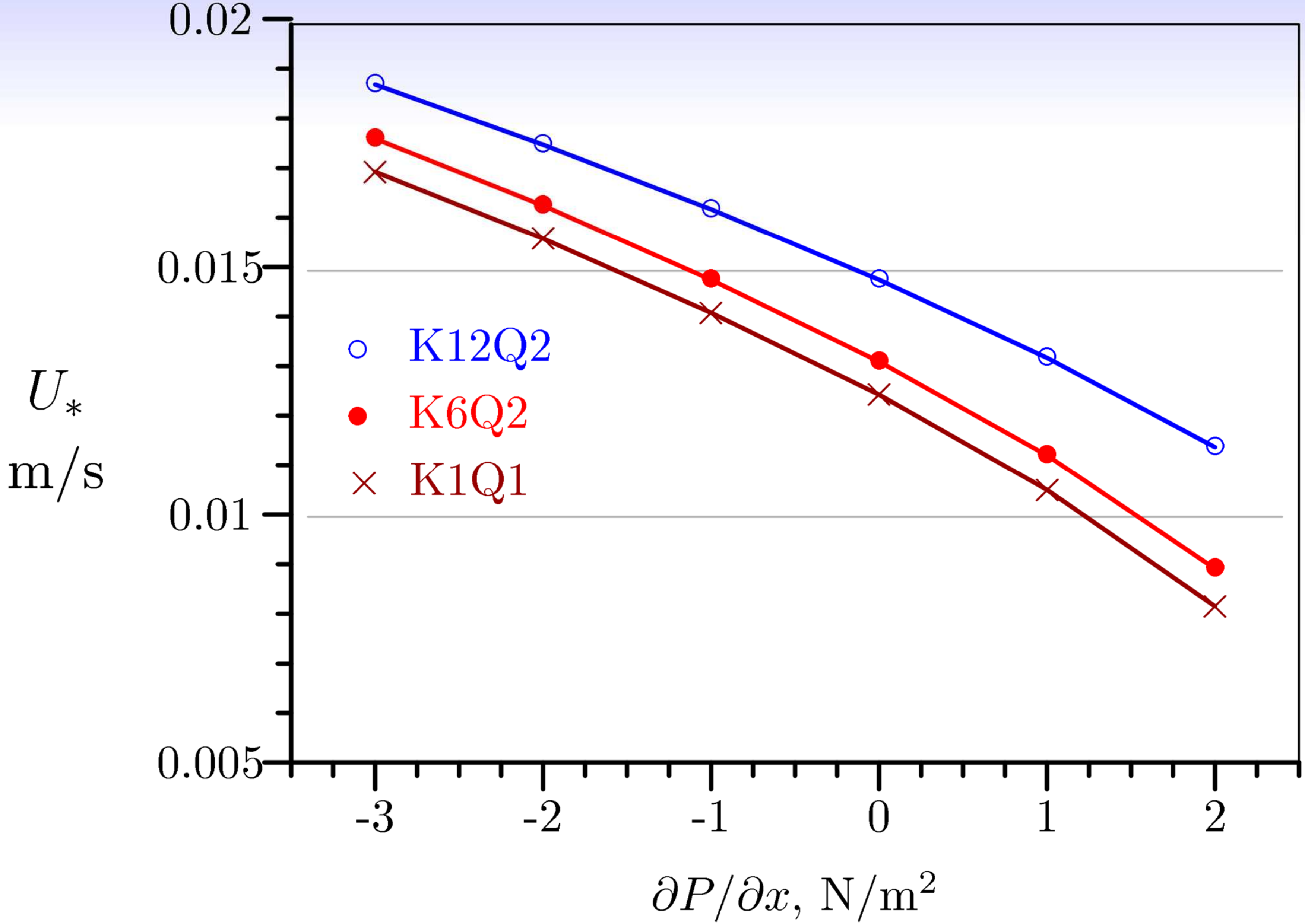
1) $\mathbf{B_n} \quad \Longrightarrow \quad \dfrac{\partial P^{(n)}}{\partial x}$

2) predict $\mathbf{B_{n+1}}$ after previous series $\left(\mathbf{B_i}, \dfrac{\partial P^{(i)}}{\partial x}\right)$

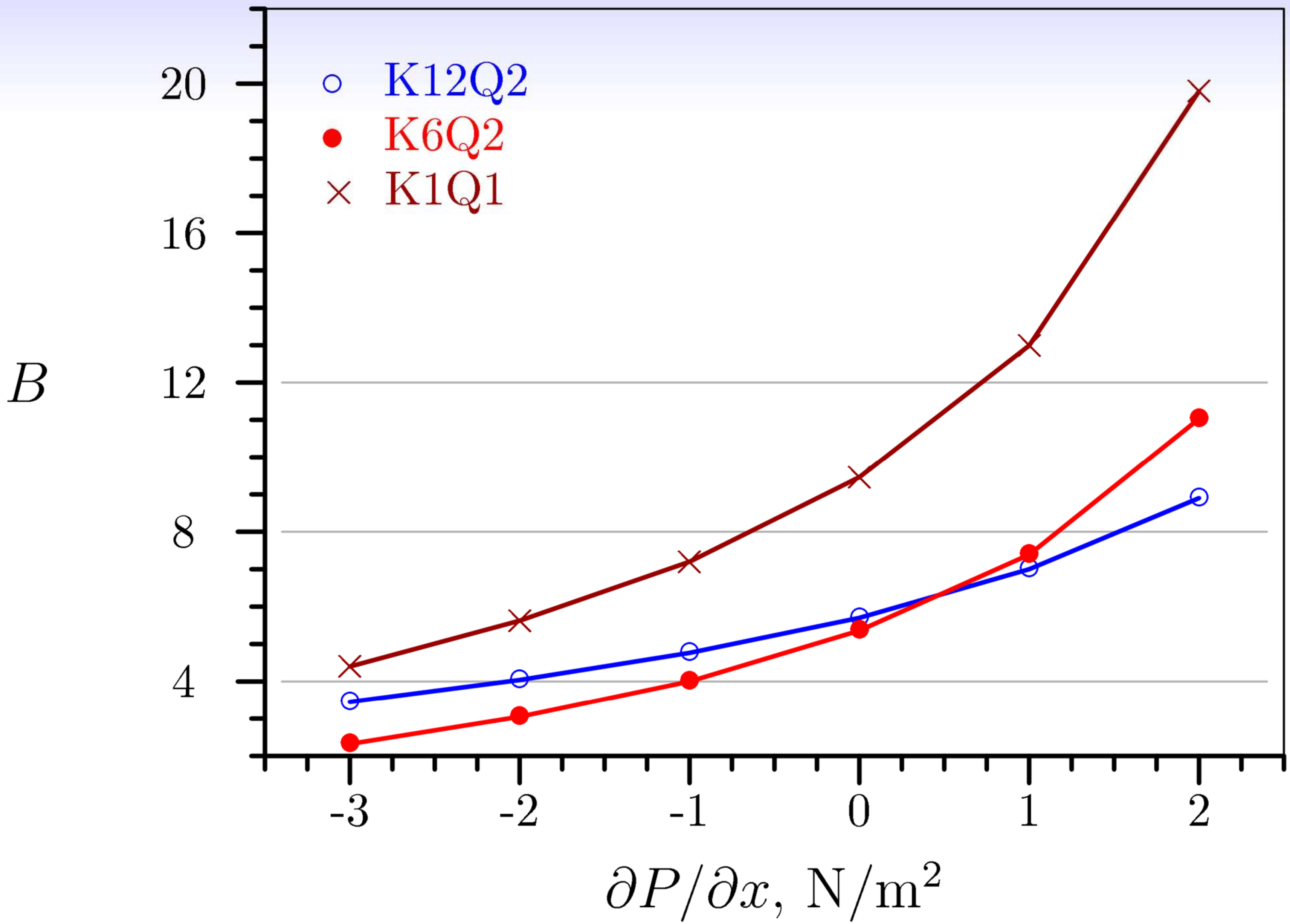
in order to get $\dfrac{\partial P}{\partial x} = a,$

3) goto 1) with $\mathbf{B_{n+1}},$

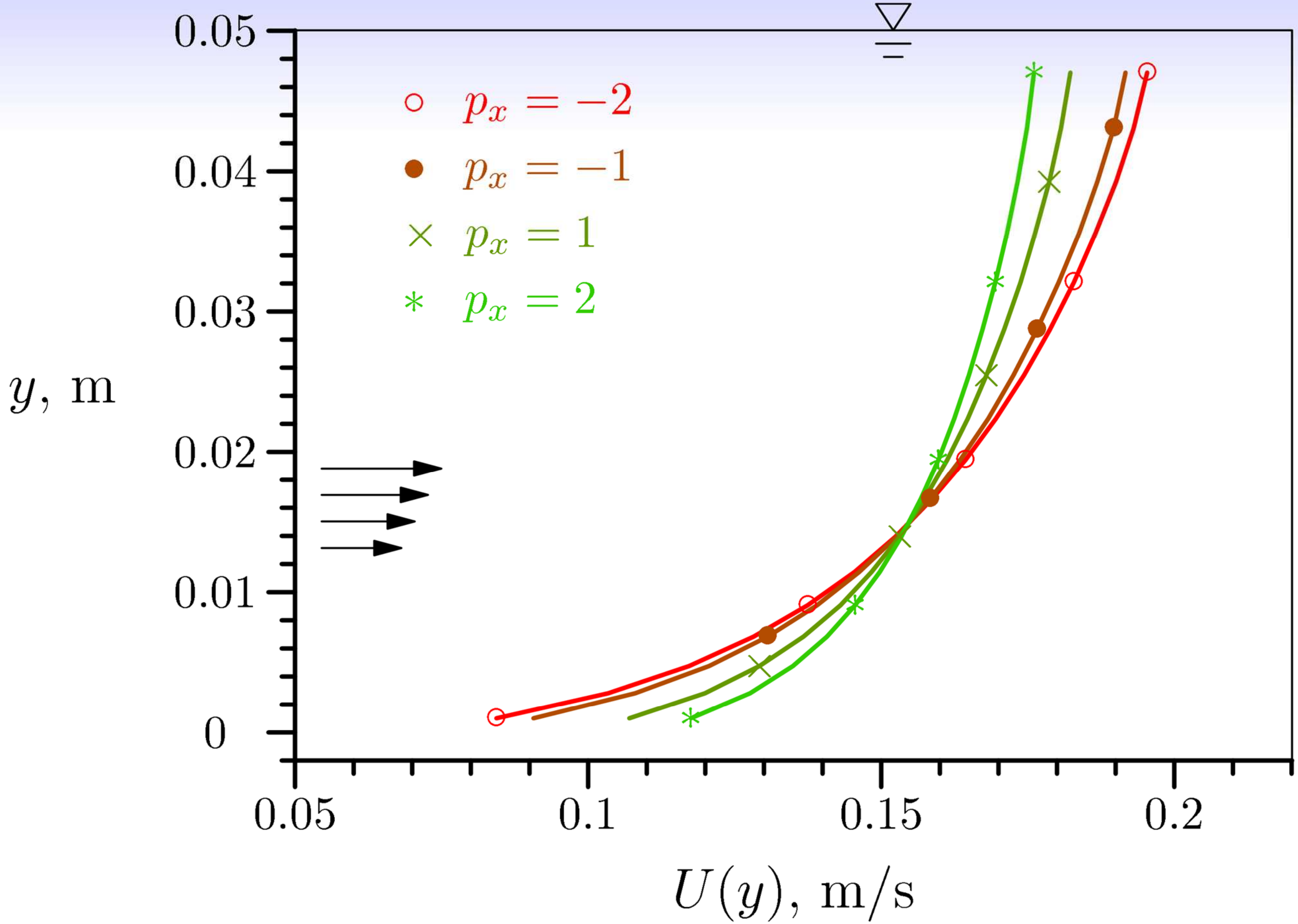
Dependence $U_* = f_1(\frac{\partial P}{\partial x})$



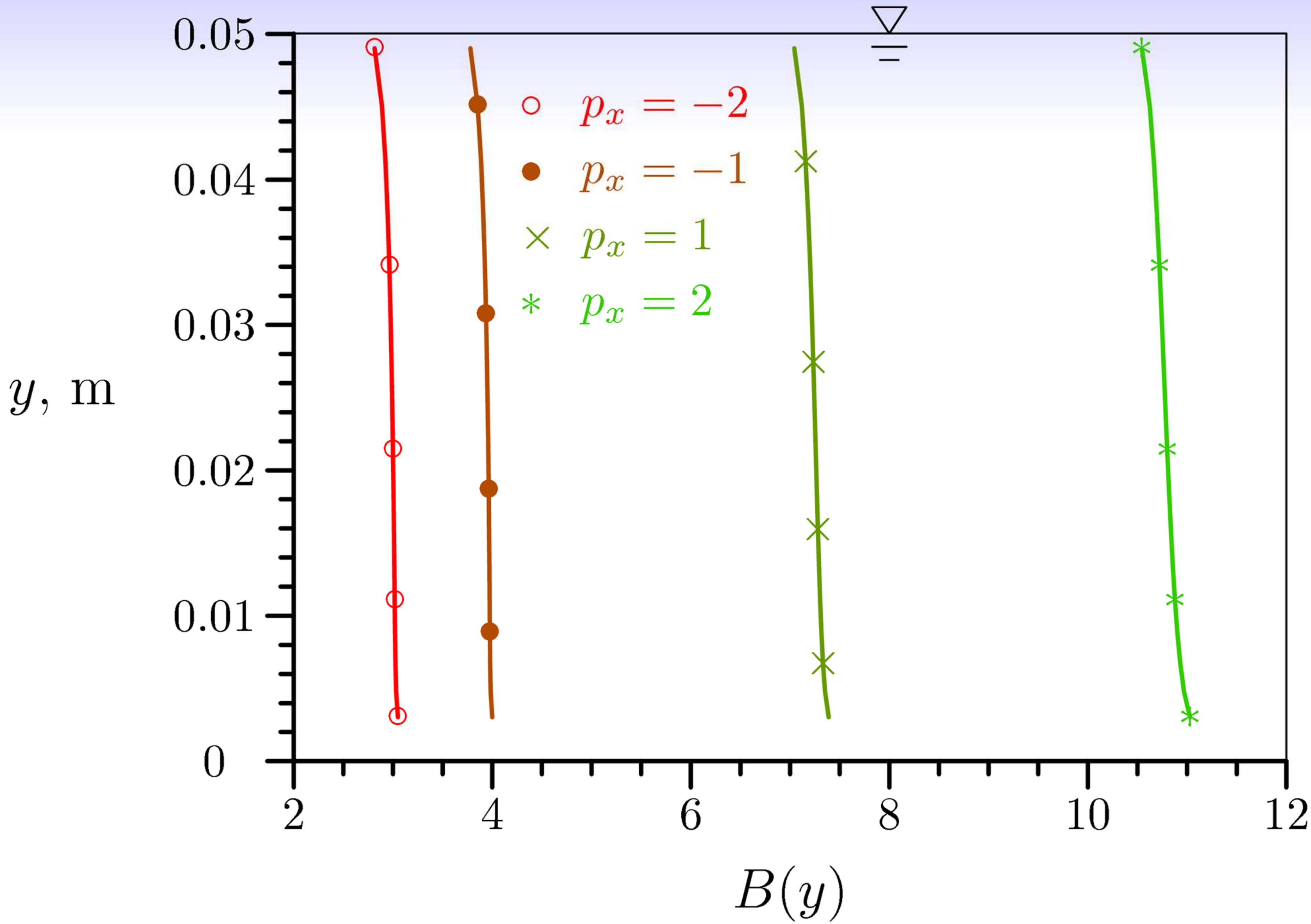
Dependence $B = f_2(\frac{\partial P}{\partial x})$



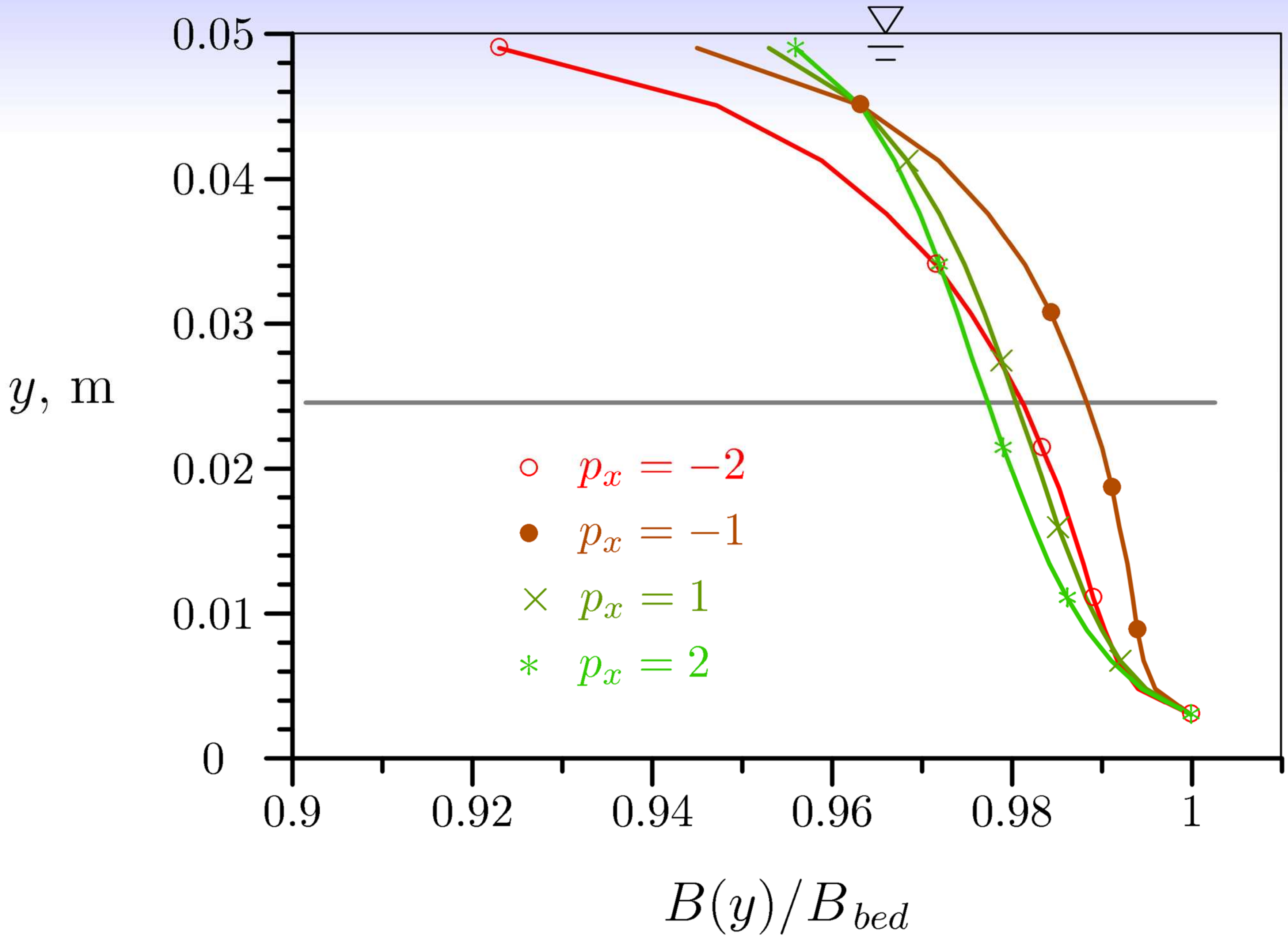
Velocity profiles $U(y)$



$B(y)$ point-wise profiles



$B(y)/B_{bed}$ profiles



Conclusions

- **rigid-lid approximation is applicable**
- **logarithmic profile is applicable for accelerated/decelerated flows**
- **inner layer (with log-law) stretches up to 0.25 of depth**
- **computational model : reasonably matches in K6, K12, and worse in K1 (relaminarization)**
- **as $\partial P/\partial x$ rises, so does B**