Where *Small is Beautiful*: mathematical modelling & free surface flows

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Outline

Introduction - in praise of simplicity and smallness

The expression *Small is Beautiful* was popularised in the title of a 1973 book "Small Is Beautiful: A Study of Economics As If People Mattered" by Ernst Friedrich Schuhmacher, a German economist and statistician who spent most of his life in Britain.

In 1974, the year after publication of the book, I went on a weekend workshop to discuss its economic and political messages in Windsor Great Park near London. It was a pleasant and optimistic era for some.

Aphorisms on simplicity

William of Ockham

A mediaeval monk and philosopher, developed the principle known as *Ockham's Razor*: if something can be explained without a further assumption, there is no reason to assume it. Any explanation should be in terms of the fewest factors or parameters.

Isaac Newton

- "Nature is pleased with simplicity, and affects not the pomp of superfluous causes",
- "Truth is ever to be found in the simplicity, and not in the multiplicity and confusion of things".

Newton did not follow his own advice re simplicity in his writing at the time, the 17C! The modern Americanised version might be better:

- "Nature is pleased with simplicity. And nature is no dummy."

Albert Einstein

- "Make things as simple as possible, but not simpler"
- "If you can't explain it simply, you don't understand it well enough."

Richard Hamming – the opening statement in a book on numerical methods:

- "The Purpose of Computing is Insight, not Numbers."

Aspects of simplicity and mathematics

Simplicity $\Leftrightarrow$ understanding: There is nothing quite like understanding – to learn, to recall, and to create, and to simplify.

Simplicity $\Leftrightarrow$ visual perception and understanding: Many scientists and engineers are highly visual, and that often goes together with (applied) mathematics, if it is simple enough.

Simplicity $\Leftrightarrow$ ease of disproof: in the sense of Karl Popper – if something is simple, it is likely to be more correct because it has withstood attempts to disprove it.

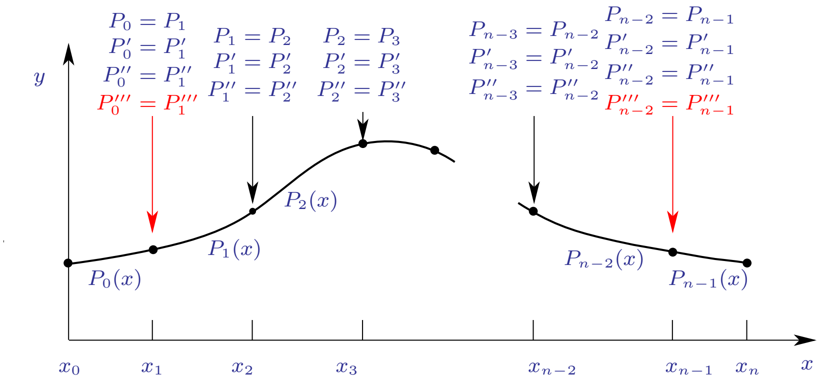
Simplicity $\Leftrightarrow$ mathematical modelling: One should make the simplest possible model and if it works, good, but if not, refine it, and repeat if necessary.

Simplicity $\Leftrightarrow$ too little or too much data: too little data leads to a simple model. Too much data often leads to complexity – use a simple model so as not to have to fit all that data.

Simplicity $\not\Leftrightarrow$ publishing of research: a disadvantage of simplicity is that reviewers are pleased to discover something that they can understand and then criticise and reject it, whereas something complicated that is beyond their understanding is likely to be accepted.

Two useful software packages to have to keep matters simple

Spline interpolation

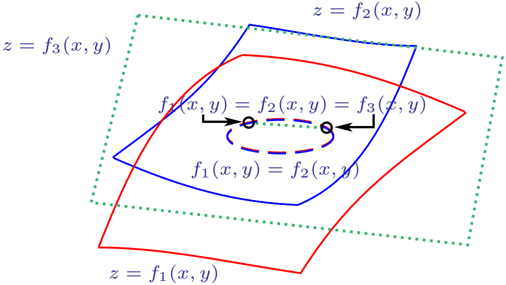


Piecewise-continuous interpolation by cubic splines with not-a-knot conditions shown in red

- Good for plotting data if it is too coarsely spaced
- Good for obtaining derivatives
- For many years spline packages required the artificial imposition of either first or second derivatives at the two end points. Much better to use a single cubic polynomial over the first two intervals and another over the last two intervals.

Solving systems of equations

- Setting up the equations for software is often complicated, for example obtaining and allocating matrix coefficients.
- In even linear problems, especially in the approximation of data, the equations are poorly conditioned and the results from linear algebra packages can be unreliable.
- If there is a mis-match between the number of equations and the number of unknowns ...
- A major problem is if the equations are nonlinear. The figure shows the near-possibility of not finding a solution at all.



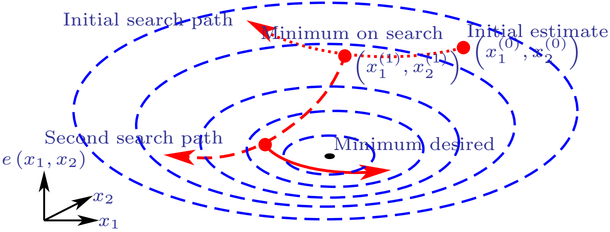
Three surfaces in (x, y, z) space and two solutions

The relative ease of solving equations by optimisation

Consider the system of equations $f_m(x_n, n = 1, \ldots, N) = 0$, for $m = 1, \ldots, M$. Optimising software takes the sum of the squares of the equations (possibly weighted by w_m) and obtains values of the unknowns x_n, by minimising e:

$$e = \sum_{m=1}^{M} w_m f_m^2,$$

Typical search procedure to find a minimum – here a function of two variables



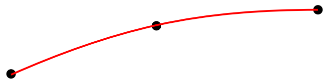
- Less need for data organisation than with linear algebra programs. One just has to write down the governing equations.
- The user can usually consider the program just as a black box.
- The methods are powerful, even for nonlinear problems. The most widely available programs are the Solver modules in spreadsheets, but there are programs in many software packages.

Interpolation and approximation

- The common problem of the representation of data, both by interpolation and approximation
- A very simple method of performing polynomial interpolation and differentiation
- Global interpolation and approximation – the reason for bad behaviour of the simplest approaches, and the general way to solve such problems

Interpolation

Apparently a simple problem:



... with a complicated solution

$$
\begin{aligned}
y &= c_0 + c_1 x + c_2 x^2, \text{ where} \\
c_0 &= \left(y_1 {x_2}^2 x_0 - y_1 {x_0}^2 x_2 - y_2 {x_1}^2 x_0 - y_0 x_1 {x_2}^2 + y_0 {x_1}^2 x_2 + y_2 {x_0}^2 x_1 \right) / D, \\
c_1 &= \left(y_2 {x_1}^2 - y_0 {x_1}^2 + y_1 {x_0}^2 - y_2 {x_0}^2 + y_0 {x_2}^2 - y_1 {x_2}^2 \right) / D, \\
c_2 &= -\left(y_2 x_1 - y_2 x_0 - y_1 x_2 + y_1 x_0 + y_0 x_2 - y_0 x_1 \right) / D, \\
D &= \left(-x_1 + x_0 \right) \left(-x_2 + x_0 \right) \left(-x_2 + x_1 \right).
\end{aligned}
$$

Now consider what it is like with N points to be interpolated ... with computational effort to solve the equations necessary $\sim N^3$ ...

Newton interpolation and Divided Differences

A much simpler solution is available. We use a *nested* form, the *Newton polynomial*:

$$p(x) = a_0 + (x - x_0)\,(a_1 + (x - x_1)\,(a_2 + \ldots (a_{N-1} + a_N\,(x - x_{N-1}))\ldots))$$

where the nth term, starting counting from zero, is a product of all the $x - x_k$, up to $k = n - 1$, and the coefficients $a_n = f\,[x_0, \ldots, x_n]$ are the *divided differences* (all starting with x_0) of the table of function values (Conte & de Boor, 1980, p42):

Divided difference table

x_i	x_0	x_1	x_2	x_3
$f[x_i]$	$a_0 = f[x_0]$	$f[x_1]$	$f[x_2]$	$f[x_3]$
$f[,]$	$a_1 = f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}$	$f[x_1, x_2] = \dfrac{f[x_2] - f[x_1]}{x_2 - x_1}$	$f[x_2, x_3] = \dfrac{f[x_3] - f[x_2]}{x_3 - x_2}$	
$f[,,]$	$a_2 = f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	$f[x_1, x_2, x_3] = \dfrac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$		
$f[,,,]$	$a_3 = f[x_0, x_1, x_2, x_3] = \dfrac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$			

OR, with initial values $a_n = f[x_n]$, for $n = 0, \ldots, N$:

for i from 1 to N do

for j from N to i step -1 do

$a_j := (a_j - a_{j-1})/(x_j - x_{j-i})$

Evaluation of function and derivatives

To evaluate the Newton interpolating polynomial

Evaluate Newton polynomial $p(X)$
$p := a_N$
for j from $N-1$ to 0 step -1 do
$\quad p := a_j + p \times (X - x_j)$

Now we express the polynomial in power form, which can be simply integrated or differentiated. There is a remarkably simple algorithm to do this. Using the notation $x_{(-1)}$ for the new centres (the notation is important for the procedure) the a_n and the x_n are all over-written where necessary:

Convert Newton polynomial to power form with centres x_{-1}
for i from 1 to N do
$\quad$ for j from $N-1$ to $i-1$ step -1 do
$\quad\quad a_j := a_j + a_{j+1} \times (x_{(-1)} - x_j)$
$\quad\quad x_j := x_{j-1}$

The a_n contain the useful information that the nth derivative at $x_{(-1)}$ is given by

$$\left.\frac{\mathrm{d}p}{\mathrm{d}x}\right|_{x_{(-1)}} = p^{(n)}\left(x_{(-1)}\right) = a_n\, n!$$

If one needs derivatives at another point, one sets x_{-1} to that and can immediately run the procedure in the last table again, and so on.

Global interpolation and approximation

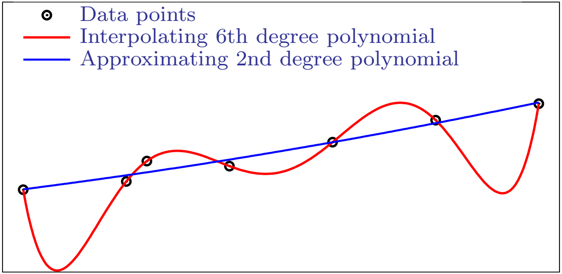
We consider a general problem, with a possibly important result. The underlying mathematics is not *Small* but our interpretations are – and are visual.

Global interpolation and approximation

We consider the approximation of a number of data points $(x_n, y_n), n = 0, \ldots, N$ by a continuous function $y = p(x)$ composed of a linear combination of a number of specified functions $p_m(x)$, each multiplied by an unknown coefficient a_m:

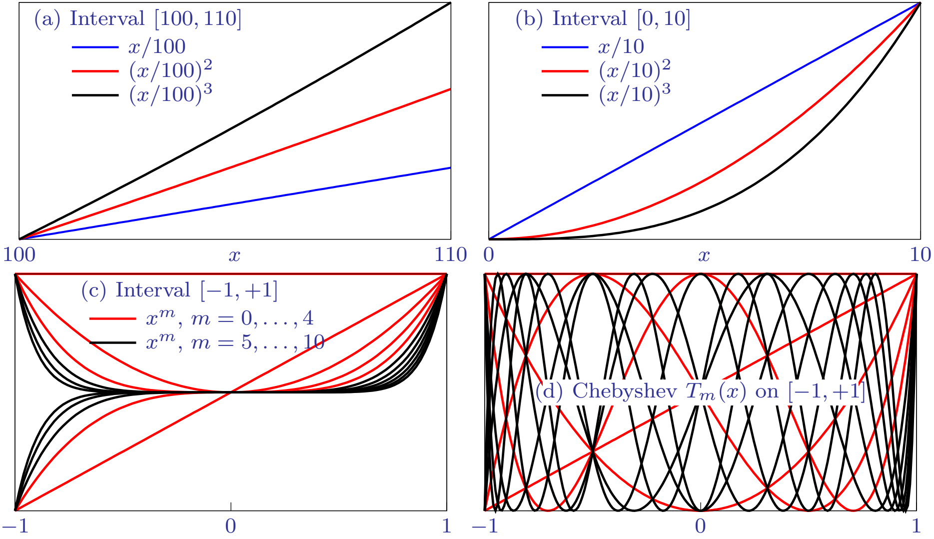
$$p(x) = \sum_{m=0}^{M} a_m p_m(x) = a_0 p_0(x) + a_1 p_1(x) + \ldots + a_M p_M(x)$$

If the number of the $M + 1$ coefficients a_m is equal to the number of data points $N + 1$ then the function passes through every data point such that it *interpolates* them. If $M < N$ then the function *approximates* the data points. In general this is to be preferred, for if we use global interpolation with $M = N$, irregularities can lead to the approximating function varying unreasonably everywhere. In Statistics this is called *overfitting*.



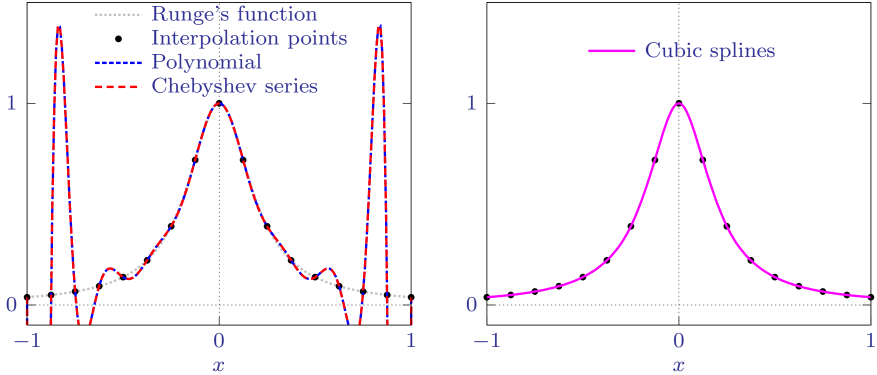
Problems with global interpolation and approximation

The simplest set of basis functions are the *monomials* $p_m(x) = x^m$. They are not very good, as they all look rather like each other for large x and for $m = 2$ or greater. Individual basis functions $p_m(x)$ should look different from each other so that irregular variation can be described efficiently.



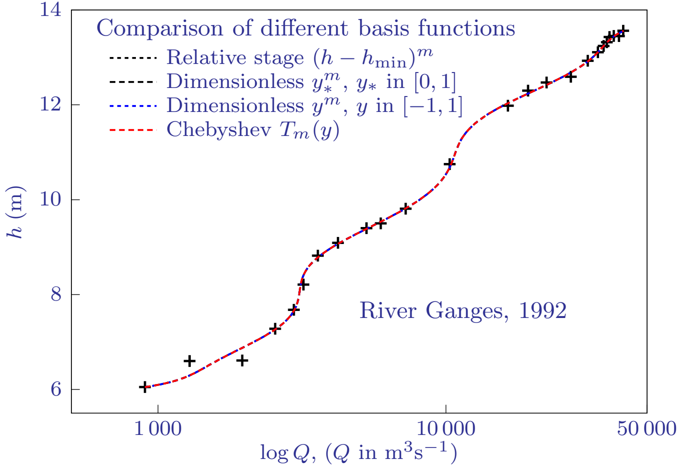
Runge's problem – global approximation with rapid local variation

- Global approximation, even by Chebyshev series, is not always the answer.
- A famous function was devised by Runge to show that to approximate the highly-curved region of rapid variation near the crest using global approximation such as here destroys the accuracy in the slowly-varying region away from the crest.
- Consider the global polynomial interpolation of a function $1/(1+25x^2)$ on $[-1,+1]$ using 17 data points, first by a 16th degree polynomial and a 16th degree Chebyshev series and then by cubic splines.



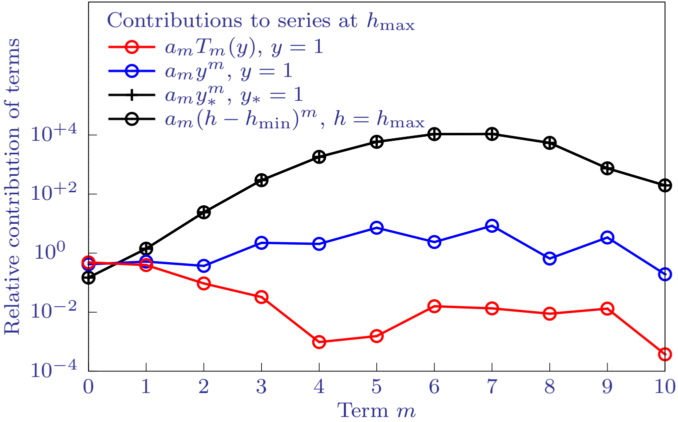
Global *approximation* of river rating data – apparently satisfactorily ...

An important quantity in river engineering is that of a *Rating curve*, one which approximates a number of measurement pairs of river height h at a gauging station and the measured flow Q. Here we apply global approximation using series of four different basis functions, *in spite of our recent experience shown in the last slides*, clearly showing coincidence, apparently satisfactorily.



… whereas the details of some of the series are completely unsatisfactory

Next figure shows the contribution of different terms in the four series at $h = h_{\max}$, which obtained the numerically coincident results in the previous figure, showing the actual huge differences in magnitude of terms in the series



For *global* approximation of data one should use orthogonal functions such as Chebyshev polynomials – or piecewise-continuous splines

Direct iteration solution of transcendental equations

Direct iteration solution of transcendental equations

- Common problem – solve an equation for x such that $f(x) = 0$, in which the function is transcendental such that it cannot be solved analytically.
- Several well-known methods for solving such equations, such as Newton's method, the Secant method, and bisection.
- Simplest method is that of *direct iteration*. In general, that method fails in about 50% of cases and numerical analysts warn against it!
- We investigate the general conditions for it to succeed, and using the knowledge so obtained we will later devise a scheme for the computation of normal depth in steady uniform flow.
- We re-arrange $f(x) = 0$ in the form $x = g(x)$, where $g(x)$ is some function of x. This gives the iteration scheme

  $$x_{m+1} = g(x_m),$$

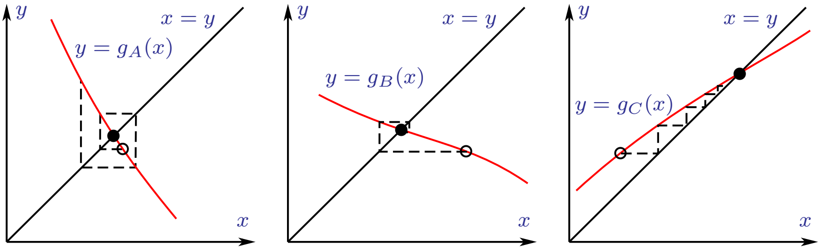
  where m is the iteration number. We assume some initial value x_0, evaluate $g(x_0)$ and use this value as the next estimate x_1 and so on.
- Conditions for which this works can be explained graphically. Solving the equation $x = g(x)$ is equivalent to solving the pair of simultaneous equations

  $$y = x, \quad \text{and} \quad y = g(x),$$

  so that if we plot the two graphs – the first is simply a straight line of gradient 1 passing through the origin, and the problem is to determine where the second graph crosses it.

Graphical interpretation of direct iteration and stability criteria

Unstable and stable behaviour of direct iteration scheme for different gradients of the function



Case A: $\left|g'_A(x)\right| > 1$, unstable   Cases B and C: $\left|g'_{B,C}(x)\right| < 1$, stable

The figures suggest the condition for convergence that the direct iteration scheme $x_{m+1} = g(x_m)$ is stable if the gradient of the curve is less than one in magnitude in the vicinity of the solution, that is,

$$\left|\frac{\mathrm{d}g}{\mathrm{d}x}(x_m)\right| < 1 \text{ for convergence.}$$

Example - carried out in two ways

Use direct iteration to solve the equation $x^3 - x - 1 = 0$

We consider two methods. The most obvious is the scheme $x_{m+1} = x_m^3 - 1$. Immediately there seems to be trouble, as the derivative of the right side ($g'(x) = 3x^2$) may well be greater than 1. We start with $x_0 = 1$, and obtain the results shown in the left table. It is obvious that the scheme is unstable. By a different rearrangement, $x_{m+1} = (x_m + 1)^{1/3}$, the right table shows that the process converges quite well.

Unstable scheme

m	x_m	$x_{m+1} = x_m^3 - 1$
0	1	$1^3 - 1 = 0$
1	0	$0^3 - 1 = -1$
2	-1	$-1^3 - 1 = -2$
3	-2	$-2^3 - 1 = -9$
4	-9	$-9^3 - 1 = -730$

Stable scheme

m	x_m	$x_{m+1} = (x_m + 1)^{1/3}$
0	1	$(1 + 1)^{1/3} = 1.259$
1	1.259	$(1.259 + 1)^{1/3} = 1.3121$
2	1.3121	$(1.3121 + 1)^{1/3} = 1.3223$
3	1.3223	$(1.3223 + 1)^{1/3} = 1.3243$
4	1.3243	$(1.3243 + 1)^{1/3} = 1.3246$

Accurate results with simple methods: Richardson & Aitken extrapolation

Accurate results with simple methods – Richardson extrapolation

- Consider the numerical value of a computational solution for some physical quantity ϕ obtained using a time or space step Δ, such that we write $\phi(\Delta)$.
- Let the computational scheme be of known nth order such that the *global* error of the scheme at any point or time is proportional to Δ^n, then if $\phi(0)$ is the exact solution, we can write the expression in terms of the error at order n:

$$\phi(\Delta) = \phi(0) + b\Delta^n + \ldots,$$

  where b is an unknown coefficient, and where the neglected terms vary like Δ^{n+1}.
- From two numerical simulations or approximations for values of Δ_1 and Δ_2 we obtain two numerical results ϕ_1 and ϕ_2. The equation for each result gives

$$\begin{aligned} \phi_1 &= \phi(0) + b\Delta_1^n + \ldots, \\ \phi_2 &= \phi(0) + b\Delta_2^n + \ldots. \end{aligned}$$

  Eliminating b between the two equations and neglecting the terms omitted, we obtain an approximation to the exact solution

$$\phi(0) \approx \frac{\phi_2 - r^n \phi_1}{1 - r^n},$$

  where $r = \Delta_2/\Delta_1$.
- The errors are now those of the higher order terms, proportional to Δ^{n+1}, so that we have gained a higher-order scheme without having to implement any more complicated numerical methods. This procedure, where n is known, is called Richardson extrapolation to the limit.

Example of Richardson extrapolation

As an example, consider the simple Euler method for solution of ordinary differential equations. For the differential equation $dy/dt = f(t, y)$, Euler's method is

$$y(t + \Delta) = y(t) + \Delta\, f(t, y(t)) + O\left(\Delta^2\right),$$

where the truncation error at a single time step Δ has been shown as $O\left(\Delta^2\right)$. In a calculation, to reach a certain point in time, the number of such steps required is proportional to $1/\Delta$, and so the global error at that point in time is $O\left(\Delta^1\right)$, and so $n = 1$ in our terminology.

Example of Richardson extrapolation

Consider the numerical solution by Euler's method of the differential equation $dy/dt = \mathrm{e}^t$ with $y(0) = 1$ as far as $t = 1$. This has the exact solution $y = \mathrm{e}^t$ and so the exact result for $y(1) = \mathrm{e} = 2.718281828$. Using first 10 steps, $\Delta_1 = 0.1$ and then 20 steps, $\Delta_2 = 0.05$, and with $n = 1$:

	Numerical result	Relative error
$N = 10$	2.805628	3.2 %
$N = 20$	2.761597	1.6 %
Richardson	2.717566	0.026 %

Accurate results with simple methods – Aitken extrapolation

Aitken's Δ^2 method can be used where we do not know the value of n, the order of the scheme. As we have one more unknown, n, we have to do a computation with a third step so that we can write a third equation in addition to those used above:

$$\phi_3 = \phi(0) + b\Delta_3^n + \ldots .$$

The equations that we now have to solve are nonlinear (the n occurs in an exponent) but they can still be solved. The b can be eliminated to give two equations

$$\frac{\phi_1 - \phi(0)}{\phi_2 - \phi(0)} = \left(\frac{\Delta_1}{\Delta_2}\right)^n \qquad \text{and} \qquad \frac{\phi_2 - \phi(0)}{\phi_3 - \phi(0)} = \left(\frac{\Delta_2}{\Delta_3}\right)^n .$$

It is possible to eliminate n between these two equations to give the single equation for $\phi(0)$. For arbitrary Δ_1, Δ_2 and Δ_3 it is a transcendental equation which cannot be solved simply. However, *if we choose the ratios of the steps to be the same*, $\Delta_1/\Delta_2 = \Delta_2/\Delta_3$, which is usually possible, the right sides of the equations are equal and we can solve for $\phi(0)$ and then for n, which is sometimes of interest, to know whether a method is converging as it should:

$$\phi(0) \approx \phi_3 - \frac{(\phi_3 - \phi_2)^2}{\phi_1 + \phi_3 - 2\phi_2}, \qquad \text{for} \quad \frac{\Delta_1}{\Delta_2} = \frac{\Delta_2}{\Delta_3}.$$

In the approach of this section, it can be seen that for a finite increase in computational effort, possibly doubling or trebling, one still has the advantage of simple schemes but with results that are more accurate by one or more orders, a handsome gain indeed.

Example of Aitken extrapolation

Apply the Trapezoidal rule to approximate the integral

$$\int_0^{\pi} \tfrac{1}{2} \sin x \, \mathrm{d}x$$

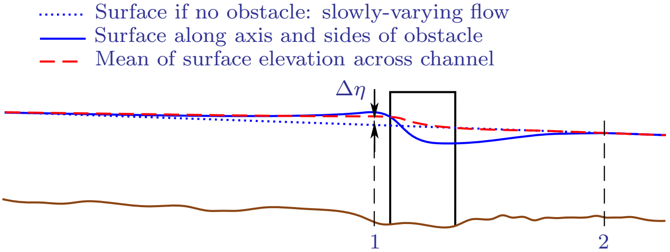
The Trapezoidal rule is of second order, $n = 2$, but we pretend neither to know this nor to use it.

	Numerical result	Relative error
$N = 8$	0.987116	1.29 %
$N = 12$	0.994282	0.57 %
$N = 18$	0.997460	0.25 %
Aitken	0.999994	0.0006 %

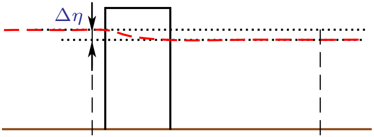
Now, calculating the value of n from one equation on the previous slide, we find $n = 2.005$, providing us with welcome evidence that we have not made a gross mistake in the programming.

Flow past an obstacle in a stream: an example of linearising a problem

Flow past a bridge pier – prototype and model



(a) The physical problem showing backwater $\Delta\eta$ just upstream of obstacle



(b) The idealised local problem, uniform channel with friction and slope effects assumed to balance

Applying conservation of momentum

The conservation of momentum principle for an obstacle in a prismatic channel, in terms of an upstream section 1 and a downstream section 2:

$$\begin{aligned} P &= \tfrac{1}{2}\rho C_{\mathrm{D}} v^2 a_1 = M_1 - M_2 \quad \text{where} \\ M &= \rho\left(gA\bar{h} + \beta Q^2/A\right) \ \text{is the momentum flux} \end{aligned}$$

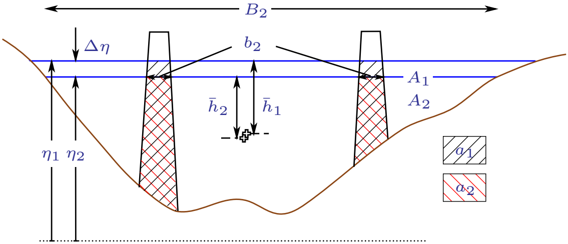
P	drag force	ρ	fluid density
C_{D}	drag coefficient	v	fluid speed impinging on the object
A	cross-sectional area	$\bar{h}$	depth of the centroid below the surface
a_1	blockage area of the object	β	Boussinesq momentum coefficient
Q	discharge	g	gravitational acceleration

We take v as being proportional to the upstream velocity $v^2 = \gamma\left(Q/A_1\right)^2$,where γ is a coefficient that recognises that the velocity which impinges on the object is not necessarily equal to the mean velocity in the flow Q/A_1. The momentum equation is then

$$\frac{1}{2}\gamma\, C_{\mathrm{D}} \frac{Q^2}{A_1^2} a_1 = \left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_1 - \left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_2 .$$

In the usual sub-critical flow, where the downstream water level is given, we want to know the effects on upstream water levels if a bridge is built. As both $A_{1,2}$ and $\overline{h}_{1,2}$ are functions of the surface elevations $\eta_{1,2}$, conditions at 2 can be evaluated, while the equation becomes a nonlinear transcendental equation for the unknown η_1, in terms shown red, which could be solved numerically with some difficulty.

Approximate linear solution



If the depth change is small, as usual, then an explicit approximate linear solution can be obtained, which reveals the nature of the problem and how the important quantities affect results. We write series for the upstream quantities at 1 in terms of those at 2.

Area

$$A_1 \approx A_2 + B_2 \Delta\eta + \ldots,$$

Blockage area

$$a_1 \approx a_2 + b_2 \Delta\eta,$$

where b_2 is the surface width of the obstacle (which for a submerged obstacle would be zero).

First moment of area about surface

$$\left(A\bar{h}\right)_1 = \left(A\bar{h}\right)_2 + A_2 \Delta\eta + O\left((\Delta\eta)^2\right)$$

Series operations

Substituting these expressions for the quantities at 1 into the momentum equation gives

$$\frac{1}{2}\gamma\, C_{\mathrm{D}}\, \frac{Q^2\left(a_2+b_2\Delta\eta\right)}{\left(A_2+B_2\Delta\eta\right)^2} = gA_2\Delta\eta + \ldots + \beta\,\frac{Q^2}{A_2+B_2\Delta\eta+\ldots} - \beta\,\frac{Q^2}{A_2}$$

We expand each side as a power series in $\Delta\eta$, neglecting terms like $\left(\Delta\eta\right)^2$. This gives a linear equation which can be solved to give an explicit approximation for the dimensionless drop across the obstacle $\Delta\eta/\left(A_2/B_2\right)$, where A_2/B_2 is the mean downstream depth:

$$\frac{\Delta\eta}{A_2/B_2} = \frac{\frac{1}{2}\,\gamma\, C_{\mathrm{D}}\, \mathsf{F}_2^2}{1-\beta\mathsf{F}_2^2}\,\frac{a_2}{A_2}$$

This explicit approximate solution has revealed the important quantities of the problem to us and how they affect the result: velocity ratio parameter γ, drag coefficient C_{D}, downstream Froude number $\mathsf{F}_2^2 = Q^2B_2/gA_2^3$, and the relative blockage area a_2/A_2.

Deductions

$$\frac{\Delta\eta}{A_2/B_2} = \frac{\frac{1}{2}\gamma\, C_{\mathrm{D}}\, \mathsf{F}_2^2}{1-\beta \mathsf{F}_2^2}\,\frac{a_2}{A_2}$$	Subcritical flow $\beta\mathsf{F}_2^2 < 1$: $\Delta\eta$ positive, surface drops Supercritical, $\beta\mathsf{F}_2^2 > 1$, $\Delta\eta$ negative, surface rises Near critical ($\beta\mathsf{F}_2^2 \approx 1$) theory not valid

Small Froude number approximation $\beta\mathsf{F}_2^2 \ll 1$ makes further deductions clearer

$$\Delta\eta \approx \frac{1}{2}\,\gamma\, C_{\mathrm{D}}\,\frac{a_2}{g A_2^3}\, Q^2$$	$\Delta\eta$ is a function of Q^2, or, Q a function of $\sqrt{\Delta\eta}$, in a manner analogous to a broad-crested weir. In numerical river models it should ideally be included as an internal boundary condition between different reaches as if it were a type of fixed control
$$\Delta\eta = \gamma\, C_{\mathrm{D}}\,\frac{(Q/A_2)^2}{2g}\,\frac{a_2}{A_2}$$	$= \gamma\, C_{\mathrm{D}} \times$ (Stagnation height of mean flow) $\times \dfrac{a_2}{A_2}$ $\gamma \approx 1$, $C_{\mathrm{D}} \approx 1$, $U_2 = Q/A_2 \approx 1\,\mathrm{ms}^{-1}$, Stagnation height $\approx$ 5cm, $a_2/A_2 \approx 10\%$, $\Delta\eta \approx$ 5mm, small in this case, BUT ...

In the Alps, for example, our calculation might be necessary

The long wave equations

- These equations – also known as the Saint-Venant equations – are based on a one-dimensional model of a free surface flow, a river, a canal, a sewer, a drain
- The main assumptions are that pressure in the water is hydrostatic, the channel is straight and motion is one-dimensional. These are all surprisingly good.
- The equations are presented and discussed. If the concept of water volume upstream of a point is introduced it can be used to simplify some operations.
- Using that concept, a single equation is obtained and linearised, to give a Telegraph equation
- That equation shows that in general "long waves" are more complicated than realised – their propagation properties depend on wave length/period
- In the case of waves that are very long, such as flood waves, a simpler nonlinear approximate equation can be obtained which is surprisingly accurate.
- Previously the linearised version of that equation and its solutions have been called "kinematic" because it has been believed that a dynamic approximation has been made. We see that that is incorrect. The equation is actually a *very long wave* or *slow-change* equation. There is no such thing as a kinematic wave.

Volume conservation equation

For theoretical purposes the equations are more useful and concise in the form using cross-sectional area A and discharge Q as dependent variables:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = i$$

where x is horizontal river space co-ordinate, t is time, and i is net inflow per unit length.

Features

- A and Q are integral quantities, characteristic of the whole section
- Remarkably, the only approximation is that the stream is assumed to be straight, otherwise it is exact
- It is linear in the dependent variables
- This allows us to introduce something very useful, the Upstream Volume V

Momentum conservation equation

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right) + \frac{gA}{B}\frac{\partial A}{\partial x} = gA\left(\tilde{S} - \Omega\, Q^2\right)$$

where

β	Boussinesq momentum coefficient	g	Gravitational acceleration
$\tilde{S}$	Mean slope at a point	Ω	Resistance coefficient

There are few parameters! It is a surprisingly simple formulation. Our only problem is to evaluate them.

Boussinesq momentum coefficient β

$\beta \approx 1.05+$ is a momentum correction factor, roughly allowing for turbulence and the velocity distribution not being constant over the section. It is poorly known; fortunately the term in which it appears is usually small and the value of the coefficient so important.

The other two parameters are occur in the two most important terms in the equation – the gravitational driving force and the resistance force.

Mean slope $\tilde{S}$

The local mean downstream slope of the stream bed evaluated across the section, defined as

$$\tilde{S} = -\frac{1}{B}\int_{B} \frac{\partial Z}{\partial x}\,\mathrm{d}y,$$

where the bed elevation is $z = Z(x, y)$, with y the transverse co-ordinate.

- If the bottom geometry is known, this can be precisely evaluated and it incorporates what in other presentations is referred to as the non-prismatic contribution
- However the geometry is likely to be only approximately known and a typical bed slope of the stream is often used

Resistance coefficient Ω

The second important contribution to horizontal momentum is that of resistance around the boundary. The term has been generalised here by using the symbol Ω:

$$\Omega(x, A) = \begin{cases} \tilde{S}/Q_{\mathrm{r}}^2, & \text{for rated discharge } Q_{\mathrm{r}}(A) \\ \Lambda P/gA^3, & \text{Chézy-Weisbach , where } \Lambda = \lambda/8 = g/C^2 \\ n^2 P^{4/3}/A^{10/3}, & \text{Gauckler-Manning} \end{cases}$$

in which Q_{r} is the steady uniform discharge for that cross-sectional area, either from a *Rating Curve*, a curve of best fit which approximates a number of points of measured discharge Q_{r} and A, or from a relationship such as Chézy-Weisbach or Gauckler-Manning giving the explicit results shown.

Misleading use of S_{f}

In many presentations of the momentum equation, the resistance term appears as $-gAS_{\mathrm{f}}$, where the symbol S_{f} does not reveal the nature of the term. It is usually described as being the slope of a line representing the variation of total head, the energy grade line. It is not. In a momentum equation it actually comes from the resistance force on the perimeter, and the symbol S_{f} is actually *the ratio of resistance force to gravitational force*. The notation has led to mistakes made in some works where S_{f} has been assumed constant, independent of Q and A.

Compound resistance

- One advantage of the Weisbach formulation, is that it is directly related to stress and force, and one can linearly superimpose force contributions, so that in a more complicated situation, where there may be bedforms, vegetation, and/or different boundary parts such as floodplains contributing to the resistance, the forces can be added and we can write, for contributions from various parts

$$\Lambda P = \sum_i \gamma_i \Lambda_i P_i$$

  The γ_i are velocity correction factors, as the relevant square of velocity for each part of a compound section is not necessarily $(Q/A)^2$.

- On the other hand, especially when the Gauckler-Manning form is used for resistance there has been much irrationality. An otherwise good recent (2002) paper lists 26 different formulae for compound or composite sections – and labelled them with the capital letters of the Latin alphabet! Almost all such formulae are nonsense.

- Some of them some just weight contributions according to individual areas A_i, some just according to perimeters P_i. Most do not include allowance for the local velocity being different from the mean of the whole section. Some do not weight different contributions at all, but combine them imaginatively.

Upstream volume – which we will find to be very useful

Consider the total volume of fluid upstream of a point x at a time t:

$$V(x,t) = \int^{x} A(x',t)\,\mathrm{d}x'$$

From simple calculus (Leibnitz):

$$A = \frac{\partial V}{\partial x}$$

The time rate of change of V at a point is equal to the total rate upstream at which the volume is increasing, which is $\int^{x} i\,dx'$, minus Q, the volume rate which is passing the point x and hence being no longer upstream. Hence,

$$\frac{\partial V}{\partial t} = -Q + \int^{x} i(x')\ dx' \quad \text{and so} \quad Q = -\frac{\partial V}{\partial t} + \int^{x} i(x')\,dx'$$

Substituting for A and Q into the volume conservation equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial x}\left(-\frac{\partial V}{\partial t} + \int^{x} i(x')\,dx'\right) = i$$

and evaluating the left side we see that the equation is satisfied identically, which the total volume of fluid should do. We have one quantity V instead of two, A and Q, and now we have only the momentum equation to satisfy.

Substituting upstream volume into the momentum equation

Substituting Q and A into the momentum equation in terms of upstream volume:

$$\frac{\partial^2 V}{\partial t^2} + 2\beta \frac{Q}{A}\frac{\partial^2 V}{\partial x \partial t} + \left(\beta \frac{Q^2}{A^2} - \frac{gA}{B(A)}\right)\frac{\partial^2 V}{\partial x^2} + gA\tilde{S}\left(1 - \left(-\frac{\partial V/\partial t}{Q_{\mathrm{r}}(A)}\right)^2\right) = 0$$

where symbols Q and A have been retained in coefficients of second derivatives.

- The momentum equation has become a second-order partial differential equation in terms of the single variable V.
- And it is unusable in this ugly form. It is more useful in theoretical works and where approximations can be made, as we now do.
- We linearise the equation by considering relatively small disturbances about a uniform flow with area A_0 and discharge Q_0. Substituting the series

  $$V = A_0 x - Q_0 t + \varepsilon v,\ A = A_0 + \varepsilon v_x,\ Q = Q_0 - \varepsilon v_t,\ \text{and}\ Q_{\mathrm{r}}(A) = Q_0 + Q'_{\mathrm{r}0}\varepsilon v_x,$$

  where εv is a small quantity, a deviation of upstream volume from that of uniform flow, $v_t = \partial v/\partial t$, $v_x = \partial v/\partial x$, and $Q'_{\mathrm{r}0} = \mathrm{d}Q_{\mathrm{r}}/\mathrm{d}A|_0$.

The Telegraph equation and the nature of long wave propagation

Performing power series operations, we obtain relatively simply, the linearised momentum equation as the *Telegraph equation*:

$$\sigma_0 \left(\frac{\partial v}{\partial t} + c_0 \frac{\partial v}{\partial x} \right) + \frac{\partial^2 v}{\partial t^2} + 2\beta U_0 \frac{\partial^2 v}{\partial x \partial t} - (C_0^2 - \beta^2 U_0^2) \frac{\partial^2 v}{\partial x^2} = 0$$

Symbols

- σ_0 – **resistance parameter / inverse time scale**: It will be found below that it is actually an important channel parameter, determining the nature of wave behaviour and computational solution properties

$$\sigma_0 = \frac{2gA_0S_0}{Q_0} = 2\frac{gS_0}{U_0} = \frac{\partial}{\partial Q}\left(gAS\frac{Q^2}{Q_{\mathrm{r}}^2} \right)\Bigg|_0$$

  It is the derivative with respect to Q of the resistance term in the momentum equation. We could argue by a rough electrical analogy that the resistance term in the momentum equation is equivalent to potential difference or voltage, while discharge Q is equivalent to current. As the derivative of voltage with respect to current gives electrical resistance, σ_0 can be thought of as a *resistance parameter* in our nonlinear case.

... the nature of long wave propagation (continued)

- c_0 – **wave speed**: This will be shown to be the speed of very long period waves.

$$c_0 = \begin{cases} \mathrm{d}Q_\mathrm{r}/\mathrm{d}A|_0\,, & \text{General expression} \\ \frac{3}{2}U_0\left(1 - \frac{1}{3}A_0 P_0'/P_0\right), & \text{Chézy-Weisbach} \\ \frac{5}{3}U_0\left(1 - \frac{2}{5}A_0 P_0'/P_0\right), & \text{Gauckler-Manning} \end{cases}$$

- $U_0 = Q_0/A_0$ – **mean fluid velocity**: used for simplicity.
- C_0 – **the speed of not-so-long waves**:

$$C_0 = \sqrt{gA_0/B_0 + \left(\beta^2 - \beta\right)U_0^2},$$

  In most textbooks this is written, not unreasonably, implicitly with $\beta = 1$ such that $C_0 = \sqrt{gA_0/B_0}$, which is usually said to be the "celerity" or "long wave speed" or "dynamic wave speed". Below it will be shown that it is actually the speed of waves only in the limit of shorter waves, but still long enough that the hydrostatic approximation holds. We call these "not-so-long" waves. They occur when waves are due to rapid gate movements. This velocity seems to be less-important than is generally believed.

We now obtain some simple solutions to the Telegraph equation in two limits.

Very long waves

- For disturbances that have a long period, such that $\partial^2/\partial t^2 \ll \sigma_0 \partial/\partial t$, which we will call "very long waves", the last three terms in the equation can be neglected, and the equation becomes the advection equation

$$\frac{\partial v}{\partial t} + c_0 \frac{\partial v}{\partial x} = 0$$

  with a general solution $v = f_1(x - c_0 t)$, where $f_1(.)$ is an arbitrary function given by the upstream conditions.
- This solution is a wave propagating downstream at speed c_0.
- The equation has been known as the "kinematic wave equation" and c_0 the "kinematic wave speed", because the approximation has previously been believed to be such that dynamic terms of order F^2 in the momentum equation have been neglected.
- Here we have shown that the only approximation has been that the wave period is long. No approximation has been made by neglecting dynamical terms. A better name is the *Very Long Wave Equation*, VLWE.

Not-so-long waves

- In the other limit, for disturbances which are shorter, such that $\partial^2/\partial t^2 \gg \sigma_0 \partial/\partial t$, for which we use the term "not-so-long" waves, the Telegraph equation becomes

$$\frac{\partial^2 v}{\partial t^2} + 2\beta U_0 \frac{\partial^2 v}{\partial x \partial t} - (C_0^2 - \beta^2 U_0^2) \frac{\partial^2 v}{\partial x^2} = 0,$$

  which is a second-order wave equation with solutions

$$v = f_{21}\left(x - (\beta U_0 + C_0)\, t\right) + f_{22}\left(x - (\beta U_0 - C_0)\, t\right)$$

  where $f_{21}(.)$ and $f_{22}(.)$ are arbitrary functions determined by boundary conditions both upstream and downstream.

- In this case the solutions are waves propagating upstream and downstream at velocities of $\beta U_0 \pm C_0$, such that in the usual terminology C_0 is the "long wave speed", and the waves travel relative to an advection velocity βU_0, where the presence of β is slightly surprising.
- We have shown here that C_0 is the speed of waves that are actually not so long, apparently paradoxically – they are long enough that the pressure distribution in the fluid is still hydrostatic, but they are short in terms of time scales given by the resistance characteristics.

Intermediate period waves

- In the general case, solutions of the long wave equations show wave propagation characteristics, velocity and rate of decay, that depend on the period of the waves, so that the waves are actually
  - diffusive – different period components decay at different rates, and
  - dispersive – different components travel at different speeds
- One can obtain solutions for the propagation behaviour in terms of wave period, but the operations are not particularly small or simple, and they are not included here.
- The widespread belief is wrong, that all waves obeying the long wave equations travel at a speed $C \approx \sqrt{gA/B}$. The behaviour is very much more complicated. There is no such thing as "a long wave speed".

The slow-change equation

The momentum conservation equation is

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta \frac{Q^2}{A}\right) + \frac{gA}{B}\frac{\partial A}{\partial x} = gA\left(\tilde{S} - \Omega\, Q^2\right)$$

For long period waves such as flood waves the terms shown pale can be neglected. Using Upstream Volume V and linearising as we did to obtain the Telegrapher equation gives the *Advection-diffusion equation*, which is very well known:

$$\frac{\partial v}{\partial t} + c_0 \frac{\partial v}{\partial x} - \frac{Q_0}{2B_0 S_0}\frac{\partial^2 v}{\partial x^2} = 0$$

We can use V even more simply without further approximation, to obtain a fully-nonlinear long period equation. Solving the remaining terms in the momentum equation for Q and using the Rated/Chézy/Manning term Q_{r} rather than Ω:

$$Q = Q_{\mathrm{r}}\sqrt{1 - \frac{1}{\tilde{S}B}\frac{\partial A}{\partial x}},$$

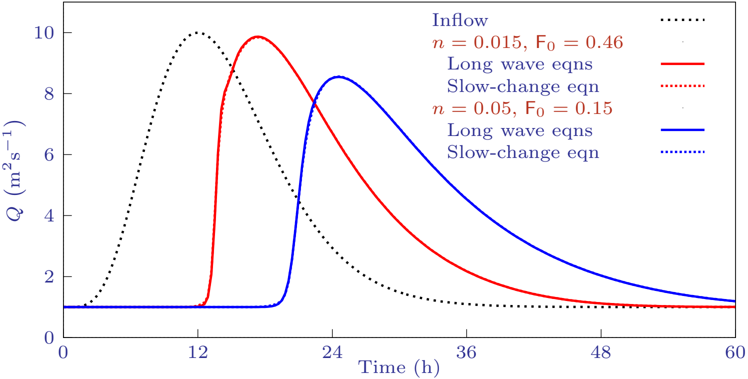
Writing both breadth B and Q_{r} as functions of area A, using $Q = -\partial V/\partial t$ and $A = \partial V/\partial x = V_x$ gives a single equation in V:

$$\frac{\partial V}{\partial t} + Q_{\mathrm{r}}\left(V_x\right)\sqrt{1 - \frac{1}{\tilde{S}B(V_x)}\frac{\partial^2 V}{\partial x^2}} = 0.$$

The only approximation relative to the long wave equations has been that the variation with time is slow, such as for flood waves.

A computational example using the slow-change routing equation

To test the hypothesis that in the very long wave and the slow change equations the *only* approximation is that motion be very long, typical of flood waves, two different boundary resistances were considered, Manning's $n = 0.015$ for a smooth boundary to give a large Froude number and $n = 0.05$ for a natural boundary. The channel was infinitely wide with a channel slope $S = 0.0005$ and the length was 50 km.



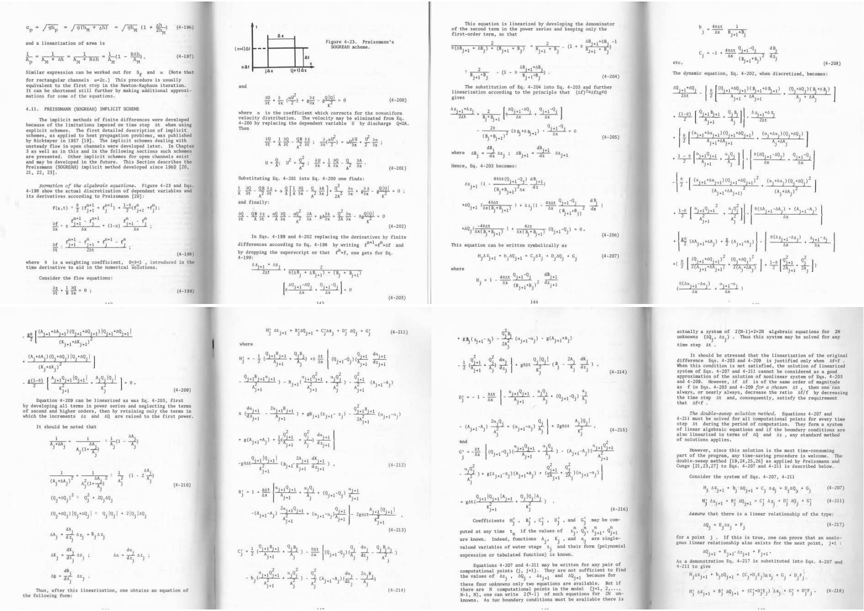
Froude number is unimportant. These are just very long wave/slow change approximations.

There is no such thing as a kinematic approximation or a kinematic wave.

Numerical solution of the long wave equations

- For fifty years it has been wrongly believed that the simplest method of numerical solution, explicit finite differences, has been unconditionally unstable
- This has led to a belief in, and reliance on, *very* complicated implicit methods
- An explicit finite difference method using divided differences is developed for arbitrary point spacing
- Boundary conditions are discussed, and it is suggested that traditional methods based on characteristics have been too complicated and of limited accuracy. Applying the finite difference method suggested here allows the general use of **F**orward-**T**ime-**Q**uadratic-**S**pace methods for advancing the solution, internally and at several different types of boundaries
- A different approach to an open downstream boundaries is proposed – simply ignoring them by using the explicit method suggested for both mass and momentum conservation.

The implicit Preissmann Box Scheme



$$c_p = \sqrt{gh_p} = \sqrt{g(h_M + \Delta h)} = \sqrt{gh_M}\,(1 + \frac{\Delta h}{2h_M}) \qquad (4\text{-}196)$$

and a linearization of area is

$$\frac{1}{A_p} = \frac{1}{A_M + \Delta A} = \frac{1}{A_M + B\Delta h} = \frac{1}{A_M}(1 - \frac{B\Delta h}{A_M}) . \qquad (4\text{-}197)$$

Similar expression can be worked out for S_f and ω (Note that for rectangular channels ω=2c.) This procedure is usually equivalent to the first step in the Newton-Raphson iteration. It can be shortened still further by making additional approximations for some of the equations.

4.11. PREISSMANN (SOGREAH) IMPLICIT SCHEME

The implicit methods of finite differences were developed because of the limitations imposed on time step Δt when using explicit schemes. The first detailed description of implicit schemes, as applied to heat propagation problems, was published by Richtmyer in 1957 [19]. The implicit schemes dealing with unsteady flow in open channels were developed later. In Chapter 3 as well as in this and in the following sections such schemes are presented. Other implicit schemes for open channels exist and may be developed in the future. This Section describes the Preissmann (SOGREAH) implicit method developed since 1960 [20, 21, 22, 23].

Formation of the algebraic equations. Figure 4-23 and Eqs. 4-198 show the actual discretization of dependent variables and its derivatives according to Preissmann [20]:

$$f(x,t) = \frac{\theta}{2}(f^{n+1}_{j+1} + f^{n+1}_{j}) + \frac{1-\theta}{2}(f^{n}_{j+1} + f^{n}_{j});$$

$$\frac{\partial f}{\partial x} = \theta\frac{f^{n+1}_{j+1} - f^{n+1}_{j}}{\Delta x} + (1-\theta)\frac{f^{n}_{j+1} - f^{n}_{j}}{\Delta x};$$

$$\frac{\partial f}{\partial t} = \frac{f^{n+1}_{j+1} - f^{n}_{j+1} + f^{n+1}_{j} - f^{n}_{j}}{2\Delta t}; \qquad (4\text{-}198)$$

where θ is a weighting coefficient, $0 \le \theta \le 1$, introduced in the time derivative to aid in the numerical solutions.

Consider the flow equations:

$$\frac{\partial z}{\partial t} + \frac{1}{B}\frac{\partial Q}{\partial x} = 0 ; \qquad (4\text{-}199)$$

Figure 4-23. Preissmann's SOGREAH scheme.

and

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}(\frac{\alpha U^2}{2}) + g\frac{\partial z}{\partial x} - g\frac{Q|Q|}{K^2} = 0 \qquad (4\text{-}200)$$

where α is the coefficient which corrects for the nonuniform velocity distribution. The velocity may be eliminated from Eq. 4-200 by replacing the dependent variable U by discharge Q=UA. Then

$$\frac{\partial U}{\partial t} = \frac{1}{A}\frac{\partial Q}{\partial t} - \frac{QB}{A^2}\frac{\partial z}{\partial t} ; \quad \frac{\partial}{\partial x}(\frac{\alpha U^2}{2}) = \alpha U\frac{\partial U}{\partial x} + \frac{U^2}{2}\frac{\partial \alpha}{\partial x} ;$$

$$U = \frac{Q}{A}; \quad U^2 = \frac{Q^2}{A^2} ; \quad \frac{\partial U}{\partial x} = \frac{1}{A}\frac{\partial Q}{\partial x} - \frac{Q}{A^2}\frac{\partial A}{\partial x} . \qquad (4\text{-}201)$$

Substituting Eq. 4-201 into Eq. 4-200 one finds:

$$\frac{1}{A}\frac{\partial Q}{\partial t} - \frac{QB}{A^2}\frac{\partial z}{\partial t} + \alpha\frac{Q}{A}\left[\frac{1}{A}\frac{\partial Q}{\partial x} - \frac{Q}{A^2}\frac{\partial A}{\partial x}\right] + \frac{Q^2}{2A^2}\frac{\partial \alpha}{\partial x} + g\frac{\partial z}{\partial x} - g\frac{Q|Q|}{K^2} = 0 ;$$

and finally:

$$\frac{\partial Q}{\partial t} - \frac{QB}{A}\frac{\partial z}{\partial t} + \frac{\alpha Q}{A}\frac{\partial Q}{\partial x} - \frac{\alpha Q^2}{A^2}\frac{\partial A}{\partial x} + gA\frac{\partial z}{\partial x} + \frac{Q^2}{2A}\frac{\partial \alpha}{\partial x} - Ag\frac{Q|Q|}{K^2} = 0 \qquad (4\text{-}202)$$

In Eqs. 4-199 and 4-202 replacing the derivatives by finite differences according to Eq. 4-198 by writing $f^{n+1}=f^n+\Delta f$ and by dropping the superscript so that f^n=f, one gets for Eq. 4-199:

$$\frac{\Delta z_{j+1} + \Delta z_j}{2\Delta t} + \frac{2}{\theta(\Delta B_j + \Delta B_{j+1}) + (B_j + B_{j+1})}\left[\theta\frac{\Delta Q_{j+1} - \Delta Q_j}{\Delta x} + \frac{Q_{j+1} - Q_j}{\Delta x}\right] = 0 \qquad (4\text{-}203)$$

This equation is linearized by developing the denominator of the second term in the power series and keeping only the first-order term, so that

$$\frac{2}{\theta(\Delta B_{j+1} + \Delta B_j) + (B_{j+1} + B_j)} = \frac{2}{B_{j+1} + B_j}\cdot(1 + \theta\frac{\Delta B_{j+1} + \Delta B_j}{B_{J+1} + B_j})^{-1}$$

$$= \frac{2}{B_{j+1} + B_j}\cdot(1 - \theta\frac{\Delta B_{j+1} + \Delta B_j}{B_{j+1} + B_j}) . \qquad (4\text{-}204)$$

The substitution of Eq. 4-204 into Eq. 4-203 and further linearization according to the principle that $(\Delta f)^2 \cong \Delta f \Delta g \cong 0$ gives

$$\frac{\Delta z_{j+1} + \Delta z_j}{2\Delta t} + \frac{2}{B_j + B_{j+1}}\left[\theta\frac{\Delta Q_{j+1} - \Delta Q_j}{\Delta x} + \frac{Q_{j+1} - Q_j}{\Delta x}\right] - \frac{2\theta}{(B_j + B_{j+1})^2}(\Delta B_j + \Delta B_{j+1})\cdot\frac{Q_{j+1} - Q_j}{\Delta x} = 0 \qquad (4\text{-}205)$$

where $\Delta B_j = \frac{dB_j}{dz}\Delta z_j$; $\Delta B_{j+1} = \frac{dB_{j+1}}{dz}\Delta z_{j+1}$

Hence, Eq. 4-203 becomes:

(4-206)

This equation can be written symbolically as

$$H_j\Delta z_{j+1} + b_j\Delta Q_{j+1} = C_j\Delta z_j + D_j\Delta Q_j + G_j \qquad (4\text{-}207)$$

where

etc. (4-208)

The dynamic equation, Eq. 4-202, when discretized, becomes:

$$- \frac{g(1-\theta)}{2}\left[\frac{A_{j+1}Q_{j+1}|Q_{j+1}|}{K^2_{j+1}} + \frac{A_jQ_j|Q_j|}{K^2_j}\right] = 0 . \qquad (4\text{-}209)$$

Equation 4-209 can be linearized as was Eq. 4-203, first by developing all terms in power series and neglecting the terms of second and higher orders, then by retaining only the terms in which the increments Δz and ΔQ are raised to the first power.

It should be noted that

$$\frac{1}{A_j + \Delta A_j} = \frac{1}{A_j(1 + \frac{\Delta A_j}{A_j})} \simeq \frac{1}{A_j}(1 - \frac{\Delta A_j}{A_j})$$

$$\frac{1}{(A_j + \Delta A_j)^2} = \frac{1}{A_j^2(1 + \frac{\Delta A_j}{A_j})^2} \simeq \frac{1}{A_j^2}(1 - 2\frac{\Delta A_j}{A_j}) \qquad (4\text{-}210)$$

$$(Q_j + \Delta Q_j)^2 \simeq Q_j^2 + 2Q_j\Delta Q_j$$

$$(Q_j + \Delta Q_j)|Q_j + \Delta Q_j| \simeq Q_j|Q_j| + 2|Q_j|\Delta Q_j$$

$$\Delta A_j = \frac{dA_j}{dz_j}\Delta z_j = B_j\Delta z_j$$

$$\Delta K_j = \frac{dK_j}{dz_j}\Delta z_j ; \qquad \Delta\alpha = \frac{d\alpha_j}{dz_j}\Delta z_j ;$$

$$\Delta B = \frac{dB_j}{dz_j}\Delta z_j .$$

Thus, after this linearization, one obtains an equation of the following form:

$$H'_j\Delta z_{j+1} + B'_j\Delta Q_{j+1} = C'_j\Delta z_j + D'_j\Delta Q_j + G'_j \qquad (4\text{-}211)$$

where

(4-212) (4-213) (4-214) (4-215) (4-216)

Coefficients H'_j, B'_j, C'_j, D'_j, and G'_j may be computed at any time t_n if the values of $z^n_j, Q^n_j, z^n_{j+1}, Q^n_{j+1}$ are known. Indeed, functions A_j, K_j, and α_j are single-valued variables of water stage z_j and their form (polynomial expression or tabulated function) is known.

Equations 4-207 and 4-211 may be written for any pair of computational points (j, j+1). They are not sufficient to find the values of Δz_j, ΔQ_j, Δz_{j+1} and ΔQ_{j+1} because for these four unknowns only two equations are available. But if there are N computational points in the model (j=1, 2,..., N-1, N), one can write 2(N-1) of such equations for 2N unknowns. As *two* boundary conditions must be available there is

actually a system of 2(N-1)+2=2N algebraic equations for 2N unknowns $(\Delta Q_j, \Delta z_j)$. Thus this system may be solved for any time step Δt.

It should be stressed that the linearization of the original difference Eqs. 4-203 and 4-209 is justified only when $\Delta f \ll f$. When this condition is not satisfied, the solution of linearized system of Eqs. 4-207 and 4-211 cannot be considered as a good approximation of the solution of nonlinear system of Eqs. 4-203 and 4-209. However, if Δf is of the same order of magnitude as f in Eqs. 4-203 and 4-209 *for a chosen* Δt, then one can always, or nearly always, decrease the ratio $\Delta f/f$ by decreasing the time step Δt and, consequently, satisfy the requirement that $\Delta f \ll f$.

The double-sweep solution method. Equations 4-207 and 4-211 must be solved for all computational points for every time step Δt during the period of computation. They form a system of linear algebraic equations and if the boundary conditions are also linearized in terms of ΔQ and Δz, any standard method of solutions applies.

However, since this solution is the most time-consuming part of the program, any time-saving procedure is welcome. The double-sweep method [19,24,25,26] as applied by Preissmann and Cunge [21,23,27] to Eqs. 4-207 and 4-211 is described below.

Consider the system of Eqs. 4-207, 4-211

$$H_j\Delta z_{j+1} + b_j\Delta Q_{j+1} = C_j\Delta z_j + D_j\Delta Q_j + G_j \qquad (4\text{-}207)$$

$$H'_j\Delta z_{j+1} + B'_j\Delta Q_{j+1} = C'_j\Delta z_j + D'_j\Delta Q_j + G'_j \qquad (4\text{-}211)$$

Assume that there is a linear relationship of the type:

$$\Delta Q_j = E_j\Delta z_j + F_j \qquad (4\text{-}217)$$

for a point j. If this is true, one can prove that an analogous linear relationship also exists for the next point, j+1 :

$$\Delta Q_{j+1} = E_{j+1}\Delta z_{j+1} + F_{j+1} .$$

As a demonstration Eq. 4-217 is substituted into Eqs. 4-207 and 4-211 to give

$$H_j\Delta z_{j+1} + b_j\Delta Q_{j+1} = (C_j + D_jE_j)\Delta z_j + G_j + D_jF_j$$

$$H'_j\Delta z_{j+1} + B'_j\Delta Q_{j+1} = (C'_j + D'_jE_j)\Delta z_j + G'_j + D'_jF_j . \qquad (4\text{-}218)$$

Numerical solution by a simple explicit method

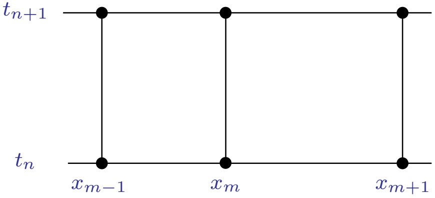
- A huge disservice to the community was made when Liggett & Cunge in 1975 wrongly calculated and stated that the simplest and most obvious finite difference numerical scheme, forward in time and central in space (FTCS) was unconditionally unstable and unfairly named it "The Unstable Scheme".
- This may have contributed to the extensive use of implicit methods throughout river hydraulics, such as the Preissmann Box scheme and a culture that complicated must be good.
- Such schemes are stable and allow large time steps, but they are very complicated and require many more calculations, including the solution at each time step of a system of nonlinear equations, the number of equations being equal to the number of space steps.
- This complexity may have contributed to computational hydraulics, once being a cottage industry with skilled people, becoming dominated by large software houses and the down-skilling of such people, similarly to the tendencies of the industrial revolution.
- A linear stability analysis shows that the scheme has a quite acceptable stability limitation, and it opens up the possibility of this as a much simpler method for computations of floods and flows in open channels – for ordinary people.

The equations and finite difference approximations

The long wave equations in terms of surface elevation η are

$$\frac{\partial \eta}{\partial t} + \frac{1}{B}\frac{\partial Q}{\partial x} = \frac{i}{B}$$
$$\frac{\partial Q}{\partial t} + 2\beta \frac{Q}{A}\frac{\partial Q}{\partial x} + \left(gA - \beta \frac{Q^2 B}{A^2} \right) \frac{\partial \eta}{\partial x} = \beta \frac{Q^2}{A^2} B \tilde{S} - gA\Omega\, Q\, |Q|$$

Generalised computational module



At any of the three computational points (x_{m+k}, t_n), $k = -1, 0, +1$ we approximate the time derivative by the forward difference approximation

$$\left.\frac{\partial f}{\partial t}\right|_{(m+k,n)} \approx \frac{f\left(x_{m+k}, t_{n+1}\right) - f\left(x_{m+k}, t_n\right)}{t_{n+1} - t_n}$$

Generalised finite difference approximations

To approximate the x derivatives at any of the points we use divided differences for the three points, numbered $m+k$, $k=-1,0,+1$, with x coordinates x_{m+k} and corresponding function values $f_{m+k}=\eta_{m+k,n}$ or $Q_{m+k,n}$. We form the divided differences

$$a_{m-1}=f_{m-1}, \quad a_m=\frac{f_m-f_{m-1}}{x_m-x_{m-1}}, \text{ and } a_{m+1}=\frac{f_{m+1}-f_m}{x_{m+1}-x_m},$$

$$\text{and immediately overwrite the last with } a_{m+1}=\frac{a_{m+1}-a_m}{x_{m+1}-x_{m-1}}$$

The general expression for the derivative at any of the three computational points is then

$$\left.\frac{\partial f}{\partial x}\right|_{m+k}=a_m+\left(2x_{m+k}-x_{m-1}-x_m\right)a_{m+1}$$

In the section below on boundary conditions, we will see how this general formulation for any one of a three-point computational module is helpful to us.

A familiar special case of the general formula is the centre difference expression for equally spaced points $x_{m-1}=x_m-\Delta x$ and $x_{m+1}=x_m+\Delta x$:

$$\left.\frac{\partial f}{\partial x}\right|_{m}=\frac{f_{m+1}-f_{m-1}}{2\Delta x}$$

Forward-time-quadratic-space (FTQS) computational scheme

The scheme becomes

$$\frac{\eta_{m+k,n+1}-\eta_{m+k,n}}{t_{n+1}-t_n} = \frac{i}{B}-\frac{1}{B}\frac{\partial Q}{\partial x}\bigg|_{m+k,n}$$

$$\frac{Q_{m+k,n+1}-Q_{m+k,n}}{t_{n+1}-t_n} = 2\beta\frac{Q}{A}\frac{\partial Q}{\partial x}+\left(gA-\beta\frac{Q^2 B}{A^2}\right)\frac{\partial\eta}{\partial x}-\beta\frac{Q^2}{A^2}B\tilde{S}+gA\Omega\,Q\,|Q|\bigg|_{m+k,n}$$

both expressions are easily re-arranged to give explicit formulae for the terms in red, the values of η and Q at point $m+k$ at the next time level $n+1$. All terms on the right are evaluated at $(m+k,n)$ as shown, each derivative $\partial\eta/\partial x$ and $\partial Q/\partial x$ at that point evaluated using the divided difference expression in terms of the three values at $m-1$, m and $m+1$.

To advance the solution for points not on the boundary, we use these two expressions with $k=0$, the centre point of our computational module.

The expressions for $k=\pm1$, the two end points of the module, will now be seen to be useful in the next section on boundary conditions.

Boundary conditions in rivers and canals

On boundary conditions as reported in the literature.

- In books on open channel hydraulics and numerical methods, there are few practical results or physical insights for finite difference boundary conditions.
- Most books that do treat the subject include an extensive discussion of the characteristic formulation of the long wave equations but provide few concrete results. Writers seem convinced that the characteristic formulation is more fundamental than the equations themselves.
- This has led to complicated presentations and paradoxically to over-simplified and less-accurate numerical approximations, because it is more difficult to approximate the characteristics to higher than first order.
- We propose to take an approach of common sense. We will *not* be making any appeal to the method of characteristics. We will be using left and right three-point differencing at boundaries, which we have already presented, without any apparent problems.
- We will introduce a new type of open downstream boundary condition which enables the accurate simulation of waves leaving a truncated computational domain.

Upstream/downstream boundaries if stage/discharge hydrograph is known

There are four cases, which can be treated in a unified symmetric/complementary manner – and all obvious after the first. We use $m = 0$ for the upstream boundary and $m = M$ for the downstream one. All use quadratic (second-order) approximation in x, more accurate than most characteristic-based methods.

Inflow $Q(x_0, t) = Q_0(t)$ known

- $Q(x_0, t_{n+1}) = Q_0(t_{n+1})$
- $\eta(x_0, t_{n+1})$: from the FTQS finite difference formula for the *mass* conservation equation, using values of Q at x_0, x_1, x_2 and t_n in the derivative term

Upstream stage $\eta(x_0, t) = \eta_0(t)$ known

- $\eta(x_0, t_{n+1}) = \eta_0(t_{n+1})$
- $Q(x_0, t_{n+1})$: from the FTQS finite difference formula for the *momentum* conservation equation, using values of η and Q at x_0, x_1, x_2 and t_n in the derivative terms

Outflow $Q(x_M, t) = Q_M(t)$ known

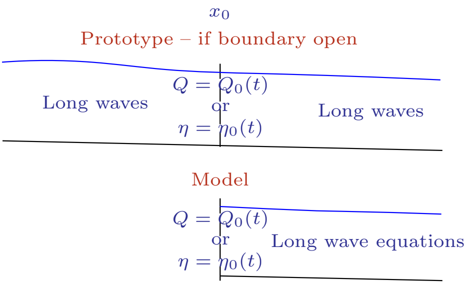
- $Q(x_M, t_{n+1}) = Q_M(t_{n+1})$
- $\eta(x_M, t_{n+1})$: from the FTQS finite difference formula for the *mass* conservation equation, using values of Q at x_{M-2}, x_{M-1}, x_M and t_n in the derivative term

Outlet stage $\eta(x_M, t) = \eta_M(t)$ known

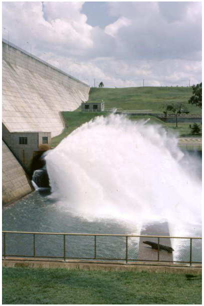
- $\eta(x_M, t_{n+1}) = \eta_M(t_{n+1})$
- $Q(x_M, t_{n+1})$: from the FTQS finite difference formula for the *momentum* conservation equation, using values of η and Q at x_{M-2}, x_{M-1}, x_M and t_n in the derivative terms

Upstream boundary – modelling not exact for an open stream

Open boundary – slight problem



Closed boundary – no problem



If the upstream boundary is at the outlet of a spillway or power station tailrace, where the stream is blocked and waves cannot propagate further upstream, then our approach seems satisfactory. If the start of the computational reach is in a channel which is not blocked, our approach is not completely correct, as any varying conditions at x_0 will also generate disturbances which will propagate upstream so that actual conditions in the downstream section near the boundary will be different from those if the channel were actually blocked there. If the input condition were an actual measured hydrograph, this problem might be cause for concern. On the other hand, any imposition of a theoretical hydrograph at such a point is arbitrary anyway, there does not seem to be a problem.

Downstream boundary - known stage-discharge relationship

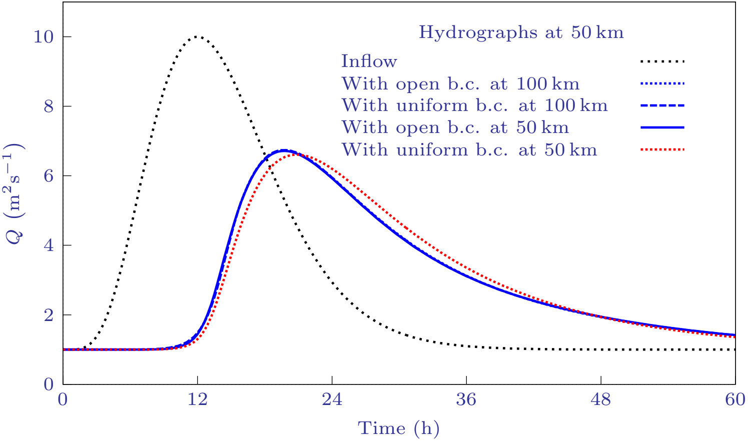
- Where there is a downstream control structure such as a spillway, weir, gate, or flume, the stage-discharge relationship $Q\left(x_M, t\right) = F\left(\eta\left(x_M, t\right)\right)$ must be known
- We assume that the $Q = F(\eta)$ relationship is not affected by unsteadiness and non-uniformity, which probably holds for relatively short control structures mentioned
- A potential difficulty – we have one equation too many: we have the FTQS finite difference formulae based on mass conservation for $\eta\left(x_M, t_{n+1}\right)$ and momentum conservation for $Q\left(x_M, t_{n+1}\right)$ *and* the relation between Q and η
- However, a sudden change in section where a typical spillway, weir, gate, or flume is placed actually violates a fundamental assumption of the long wave momentum equation, that variation in the channel is long. We can easily ignore that equation near such a sudden change
- Fortunately, the mass conservation equation, is still valid near a sudden change – it requires only the assumption that water surface is horizontal across the channel (and that the channel is straight)
- Therefore, the procedure is: we obtain the updated value $\eta\left(x_M, t_{n+1}\right)$ from the FTQS finite difference formula for the *mass conservation* equation, using values of Q at x_{M-2}, x_{M-1}, x_M and t_n in the derivative term and then we use the *stage-discharge relationship* to calculate $Q\left(x_M, t_{n+1}\right) = F\left(\eta\left(x_M, t_{n+1}\right)\right)$

Open downstream boundary

- A common boundary is where the computational domain is truncated at some point in the stream. The HEC-RAS manual calls this the *Normal Depth* boundary and uses Manning's equation to give a stage considered to be normal depth if uniform flow conditions existed downstream with that discharge.
- Because that is not correct, the manual suggests that the computational domain be artificially extended and this boundary condition be used far enough downstream from the study area that it does not affect the results there.
- However, if one can truncate a computational domain then it must be because downstream the region is unimportant and no significant information is coming back from that region.
- We advocate simply doing away with the downstream boundary condition if it is wrong or arbitrarily approximated. Instead we suggest simply treating the end point as if it were an ordinary point in the stream and using *both* FTQS formulae for $\eta\left(x_M, t_{n+1}\right)$ and $Q\left(x_M, t_{n+1}\right)$ there, with the three-point leftwards approximations for the last point x_M in terms of values at x_{M-2}, x_{M-1}, and x_M.
- For streams on small slopes, where downstream effects diffuse upstream more, this may be less accurate, but in the example below we show that that was not noticeable.

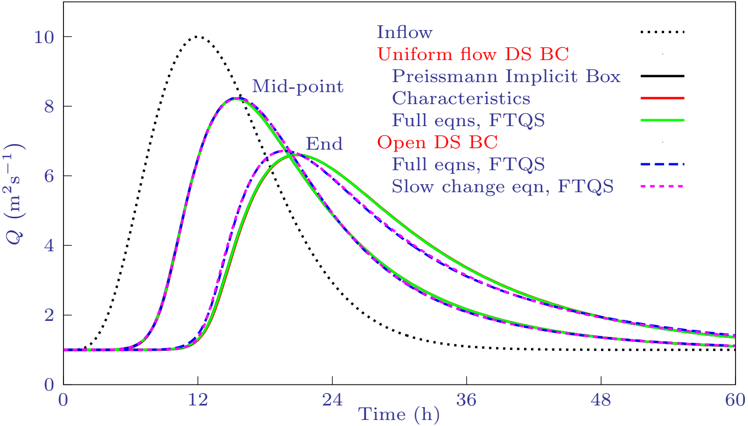
Testing the open downstream boundary condition

Test case: a relatively flat stream of slope 0.0001 where downstream boundary effects are likely to be more important, length 100 km, of infinite width, so all results are per unit width, with a Manning $n = 0.05$. An initial flow of $q = 1\,\mathrm{m^2 s^{-1}}$ was increased smoothly to $q = 10\,\mathrm{m^2 s^{-1}}$ and back down to the original flow using a model inflow hydrograph



Testing the performance of the FTQS schemes

- Simulations for the same stream as used for testing the open boundary condition
- The first three methods considered all used a uniform flow downstream (DS) boundary condition (BC). The Preissmann Implicit Box scheme, the Method of Characteristics (actually with quadratic variation), and the FTQS finite difference scheme all agreed closely with each other at both the mid-point of the computations and at the downstream boundary
- Open boundary condition at the downstream point – the FTQS finite difference scheme and the simpler slow-change equation to be presented below



The presentation at the 36th International School of Hydraulics at Jachranka near Warsaw in May 2017 ended here – the speaker ran out of time!

A critique of Muskingum flood routing

Muskingum methods for flood routing

- Muskingum flood routing is an explicit method which has been used extensively for many years
- There is empirical evidence that it can give poor results. Here we show why.
- Classical Muskingum approach: the total mean volume transport rate is assumed equal to the discharge at some intermediate point in the reach:

$$\frac{\mathcal{V}}{\Delta t} = \frac{\mathcal{V}}{\Delta x / c} = (1 - \Theta)\, I + \Theta O$$

  where $\mathcal{V}$ is reach storage volume, $\Delta t = \Delta x / c$ is the time taken for the volume to pass through the reach, of length Δx, c is wave speed, I is inflow, O is outflow, and Θ is a dimensionless distance which is 0 at the initial section and 1 at the end.
- Equating the mean volume flow rate to that at a particular section (a certain value of Θ) is an irrational, albeit plausible, approximation. There is no reason for it to be equal to the value at any particular point: the physical processes at work are more complicated than that, effectively including diffusion.

Numerical method

The exact mass conservation equation for a finite reach:

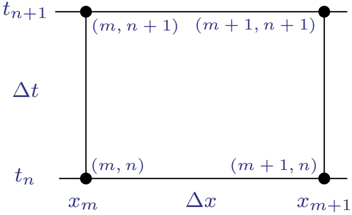
$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t} = I - O$$

Muskingum methods approximate this differential equation between time levels 1 and 2 by a forward difference for the derivative and a trapezoidal approximation for the right side:

$$\frac{\mathcal{V}_2 - \mathcal{V}_1}{\Delta t} = \frac{I_1 + I_2}{2} - \frac{O_1 + O_2}{2}$$

Substituting the Muskingum approximation for $\mathcal{V}$ as a function of I and O shown on the previous slide

$$\frac{\Delta x}{c\Delta t}\left(\left(\left(1-\Theta\right) I_2 + \Theta O_2\right) - \left(\left(1-\Theta\right) I_1 + \Theta O_1\right)\right) = \frac{I_1 + I_2}{2} - \frac{O_1 + O_2}{2}.$$



And introducing the more general computational notation with Q as discharge, m as space index, n as time index, and setting $I_1 = Q_{m,n}$, $I_2 = Q_{m,n+1}$, $O_1 = Q_{m+1,n}$, $O_2 = Q_{m+1,n+1}$ gives the Muskingum finite difference equation:

Muskingum formula and consistency analysis

$$\begin{aligned}(-2\Delta x\,(1-\Theta)-c\Delta t)\,Q_{m,n}+(-2\Delta x\Theta+c\Delta t)\,Q_{m+1,n}\\ +(2\Delta x\,(1-\Theta)-c\Delta t)\,Q_{m,n+1}+(2\Delta x\Theta+c\Delta t)\,Q_{m+1,n+1}=0\end{aligned}$$

This can be re-arranged to give an explicit expression for $Q_{m+1,n+1}$, the value at the downstream point at the later time.

Consistency analysis – the actual differential equation being solved

We write bi-dimensional Taylor series about point (m,n) for the three values $Q_{m+1,n}$, $Q_{m,n+1}$, and $Q_{m+1,n+1}$,, the latter being the most general case, for example,

$$\begin{aligned}Q_{m+1,n+1}=Q_{m,n}+\Delta x\left.\frac{\partial Q}{\partial x}\right|_{m,n}+\Delta t\left.\frac{\partial Q}{\partial t}\right|_{m,n}\\ +\tfrac{1}{2}\left(\Delta x^2\left.\frac{\partial^2 Q}{\partial x^2}\right|_{m,n}+2\Delta x\Delta t\left.\frac{\partial^2 Q}{\partial x\partial t}\right|_{m,n}+\Delta t^2\left.\frac{\partial^2 Q}{\partial t^2}\right|_{m,n}\right)+\dots\end{aligned}$$

After substitution of such a series for each computational point into the formula and dropping the (m,n) subscripts, we find that the Muskingum equation corresponds to the differential equation

$$\frac{\partial Q}{\partial t}+c\frac{\partial Q}{\partial x}+\frac{c\Delta x}{2}\frac{\partial^2 Q}{\partial x^2}+\left(\frac{c\Delta t}{2}+\Delta x\,\Theta\right)\frac{\partial^2 Q}{\partial x\partial t}+\frac{\Delta t}{2}\frac{\partial^2 Q}{\partial t^2}=O\left(\Delta x^2,\Delta x\Delta t,\Delta t^2\right)$$

Consistency analysis (continued)

However, it is known that a good approximation to the long wave equations is

$$\frac{\partial Q}{\partial t} + c\frac{\partial Q}{\partial x} - \nu\frac{\partial^2 Q}{\partial x^2} = 0 \qquad \text{(Advection-Diffusion Equation)}$$

containing only a single second derivative $\partial^2 Q/\partial x^2$, with diffusion coefficient $\nu = Q/\left(2BS\right)$, where B is surface width of channel and S is slope. To eliminate the time derivatives in the second derivatives on the previous slide, we write the advection-diffusion equation to first order

$$\frac{\partial Q}{\partial t} + c\frac{\partial Q}{\partial x} \approx 0,$$

and use this to replace $\partial/\partial t$ in the previous second derivatives by $-c\,\partial/\partial x$.The resulting equation is

$$\frac{\partial Q}{\partial t} + c\frac{\partial Q}{\partial x} = c\Delta x\left(\Theta - \tfrac{1}{2}\right)\frac{\partial^2 Q}{\partial x^2} + O\left(\Delta x^2, \Delta x\Delta t, \Delta t^2\right)$$

However the actual physical diffusion is $\nu = Q/\left(2BS\right)$, and so if we set

$$\Theta = \tfrac{1}{2} + \frac{\nu}{c\Delta x} = \tfrac{1}{2} + \frac{Q}{2BSc\Delta x}$$

we would obtain the advection diffusion equation with the correct physical diffusion.

Consistency analysis (continued)

However, there is a problem. As Δx is in the denominator of this definition of Θ, the approach described, of performing the consistency analysis using series operations in Δx and Δt, and then obtaining the value of Θ such that the computational diffusion equals the physical diffusion, is not correct.
To determine what differential equation is actually being solved, we substitute the value of Θ obtained and *then* substituting the Taylor series and performing series operations. We find that the differential equation that Muskingum routing actually solves is

$$\frac{\partial Q}{\partial t} + c\frac{\partial Q}{\partial x} + \frac{\nu}{c}\frac{\partial^2 Q}{\partial x \partial t} = O\left(\Delta x, \Delta t\right)$$

This has a mixed second derivative such that it is not the advection-diffusion equation. We can see how this relates to the desired advection-diffusion equation by using the first two terms to write $\partial/\partial t = -c\,\partial/\partial x + O\left(\nu\right)$ and to use this in the second derivative term to give an approximate version of the equation that is being approximated:

$$\frac{\partial Q}{\partial t} + c\frac{\partial Q}{\partial x} - \nu\frac{\partial^2 Q}{\partial x^2} = O\left(\nu, \Delta x, \Delta t\right),$$

which is the advection-diffusion equation, but where the error terms on the right include the diffusion ν. The Muskingum approximation is accurate only for small diffusion – not for channels of small slope.

Steady flow in rivers and canals

- The special and important case of the long wave equations for discharge that is constant in time and space
- The equations and their nature
- Traditional textbook methods – one unnecessarily complicated and one wrong
- Simple explicit methods
- An approximate linearised mathematical model of steady flow in a river

The problem

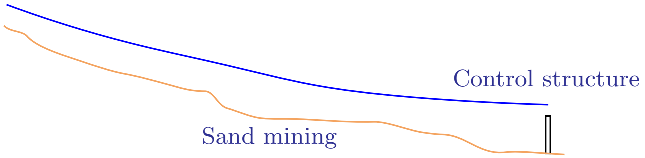


Figure: Typical gradually-varied flow problems – how far does the influence of the control structure extend upstream and/or what is the effect of sand mining?

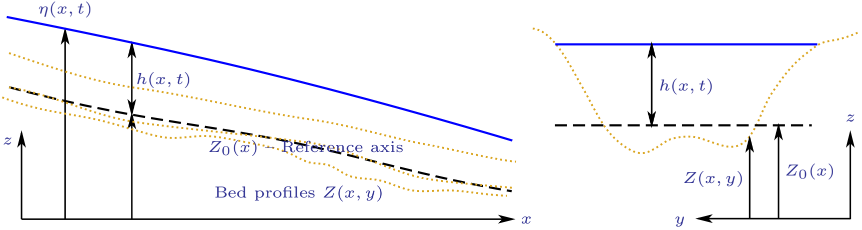
Gradually-varied flow equation – GVFE

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = \frac{\tilde{S}\beta\mathsf{F}^2 - \Omega Q^2}{1-\beta\mathsf{F}^2} \left(\approx -\Omega Q^2 \text{ for } \mathsf{F}^2 \text{ small, the common case}\right)$$

η:	Free surface elevation	$\tilde{S}$:	Mean slope at a section
β:	Momentum coefficient	F^2:	Froude number, Q^2B/gA^3
Ω:	Resistance factor		

$$\Omega(x,h) = \begin{cases} \tilde{S}/Q_{\mathrm{r}}^2, & \text{for rated discharge } Q_{\mathrm{r}}(h) \\ \Lambda P/gA^3, & \text{Chézy-Weisbach} \\ n^2P^{4/3}/A^{10/3}, & \text{Gauckler-Manning.} \end{cases}$$

Using a depth-like quantity h which we pretend we know



The tradition is not to use η, but instead a depth-like quantity $h = \eta - Z_0$, where Z_0 is the elevation of a longitudinal axis, almost always the supposed bed of the channel. The GVFE becomes

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S_0 + \beta\left(\tilde{S} - S_0\right)\mathsf{F}^2 - \Omega Q^2}{1 - \beta \mathsf{F}^2},$$

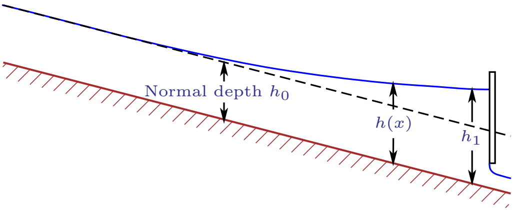
where $S_0 = -\mathrm{d}Z_0/\mathrm{d}x$, the slope of the reference axis, positive downwards. We almost never know the details of $\tilde{S}$ so here we assume that $\tilde{S} = S_0$, which we now write as S, giving

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S - \Omega Q^2}{1 - \beta \mathsf{F}^2}$$

where in general both Ω and F are functions of both x and h, while in a prismatic channel, functions just of h.

Problems using h

Because of our use of h, we pretend that we know the bed in great detail, or, that our channel looks like this:



This shows a typical subcritical flow retarded by a structure, showing the free surface disturbance decaying upstream, and if the channel is prismatic, to constant normal depth.

Traditional textbook methods – each with problems

The "Standard" step method

The almost trivial energy derivation, ignoring non-prismatic effects, is that the rate of change of total head H is given by the empirical expression for the energy gradient

$$\frac{\mathrm{d}H}{\mathrm{d}x} = -\Omega(x,h)Q^2 \quad \text{where} \quad H = Z_0(x) + h + \alpha\frac{Q^2}{2gA^2(x,h)}$$

The computational approximation scheme is

$$\frac{H_{i+1}\left(h_{i+1}\right) - H_i\left(h_i\right)}{x_{i+1} - x_i} = -\tfrac{1}{2}Q^2\left(\Omega\left(x_i, h_i\right) + \Omega\left(x_{i+1}, h_{i+1}\right)\right)$$

- $H(h)$ and $\Omega(x,h)$ are both complicated geometrical functions of h
- Requires numerical solution of a transcendental equation at each time step.
- The method advocated by Chow in 1959, in a pre-computer era.

USA, c. 1960

Using slide rules, possibly simulating channel flow

Russia, c. 1960

"Calculators" simulating the whole atmosphere, following L. F. Richardson

The “Direct” step method – distance calculated from depth

- Applied by taking steps in the water depth and calculating the corresponding step in x.
- It has some advantages, in that iterative methods are not necessary (“Direct”).
- Practical disadvantages are:
  - It is applicable only to prismatic sections
  - Results are not obtained at specified points in x
  - As uniform flow is approached the steps become infinitely large
  - AND, it is wrong, as we now show

Consider the “specific head”, the head relative to the local channel bottom, denoted here by H_0:

$$H_0(h) = H(h) - Z = h + \alpha \frac{Q^2}{2gA^2(h)}.$$

The differential equation becomes, after inverting each side

$$\frac{\mathrm{d}x}{\mathrm{d}H_0(h)} = \frac{1}{S - \Omega Q^2}.$$

A mistake and a correction

- The differential equation is now approximated, the left side by a finite difference expression $(x_i - x_{i+1}) \,/\, (H_{0,i} - H_{0,i+1})$.
- For the right side the numerical method as set out in textbooks is to take the mean of just the *denominator* at beginning and end points, and so to write

$$x_{i+1} = x_i + \frac{H_{0,i+1} - H_{0,i}}{\frac{1}{2}\left(S_i + S_{i+1} - Q^2\left(\Omega_i + \Omega_{i+1}\right)\right)}$$

  where the red shows the quantity that is a supposed mean value.
- While this is a plausible approximation, it is not mathematically consistent. What should be done is to use the mean value of the whole right side of the differential equation at beginning and end points, to give a trapezoidal approximation of the right side, which leads to

$$x_{i+1} = x_i + \left(H_{0,i+1} - H_{0,i}\right)\,\frac{1}{2}\left(\frac{1}{S_i - Q^2\Omega_i} + \frac{1}{S_{i+1} - Q^2\Omega_{i+1}}\right).$$

Standard simple numerical methods for differential equations - 1

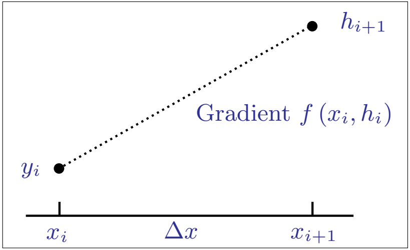
Two simple methods: Euler and Heun

We write the differential equation as

$$\frac{\mathrm{d}h}{\mathrm{d}x} = f\,(x,h) = \frac{S(x) - \Omega\,(x,h)\,Q^2}{1 - \beta \mathsf{F}^2\,(x,h)}$$

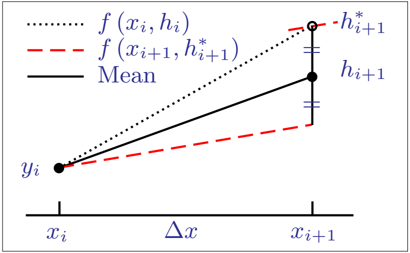
The two simplest methods are:

Euler



$$h_{i+1} \approx h_i + \Delta x\, f\,(x_i, h_i) + O\,(\Delta x)^2$$

Heun



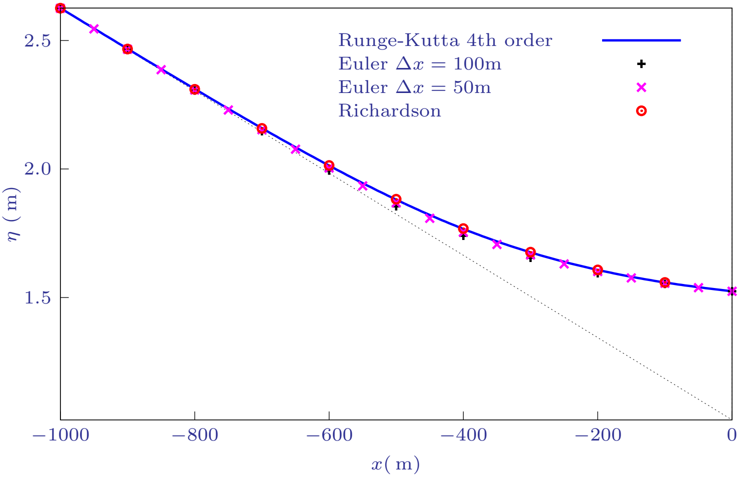
$$\begin{aligned} h^*_{i+1} &\approx h_i + \Delta x\, f\,(x_i, h_i)\,, \\ h_{i+1} &\approx h_i + \frac{\Delta x}{2} \times \\ &\quad \left(f\,(x_i, h_i) + f\,(x_{i+1}, h^*_{i+1})\right) \\ &\quad + O\,(\Delta x)^3 \end{aligned}$$

Standard simple numerical methods for differential equations

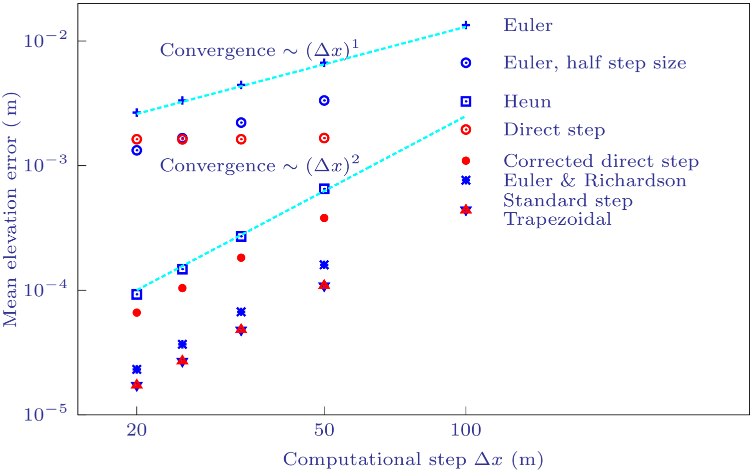
- Euler's method is the simplest but least accurate – yet it might be appropriate for open channel problems where quantities may only be known approximately
- One can use simple modifications such as Heun's method to gain better accuracy, or use Richardson extrapolation, or even more simply, just take smaller steps Δx
- For greater accuracy one can use the Trapezoidal method, simply repeating the second Heun step several times, setting $h_{i+1}^* = h_{i+1}$ each time
- Often these two methods are not presented in hydraulics textbooks as alternatives, yet they are simple and flexible, and reveal the nature of what we are doing
- The step Δx can be varied at will, to suit possible irregularly spaced cross-sectional data
- In many situations, where $\mathsf{F}^2 \ll 1$, we can ignore the $\beta\mathsf{F}^2$ term in the denominators, giving a notationally simpler scheme

Comparison of schemes

Example 10-1 of Chow (1959) was solved using: a flow of $11.33\,\mathrm{m^3s^{-1}}$ passes down a trapezoidal channel of gradient $S = 0.0016$, bed width $6.10\,\mathrm{m}$ and channel side slopes $V\!:\!H = 0.5$, the quantity α or $\beta = 1.1$, and Manning's $n = 0.025$. At $x = 0$ the flow is backed up to a depth of $1.524\,\mathrm{m}$. The backwater curve was computed for $1000\,\mathrm{m}$ in 10 steps and then 20 for Richardson extrapolation.



Convergence of numerical schemes



- Using Euler, then applying Richardson extrapolation, gave the third most accurate of all the methods, more than enough for practical purposes
- The most accurate were the Standard step method and the Trapezoidal method
- There *is* something wrong with the conventional Direct step method as we have suggested, while the corrected scheme is highly accurate

A mathematical model of steady flow in a river

- Often the precise details of a stream are not known, and it is quite legitimate to make approximations
- These might give us more insight and understanding of the problem
- Now a model is made where the GVFE is linearised and a general solution obtained
- Simple deductions as to the length of backwater effects can be made
- One can calculate an approximate solution for a whole stream if the variation in the resistance coefficient and geometry are known or can be estimated
- There is more of a balance between what we know (usually little) and the (un)sophistication of the model

The GVFE is

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S - \Omega(x,h)Q^2}{1 - \beta \mathsf{F}^2(x,h)}$$

We consider small perturbations about an underlying uniform flow of slope S_0 and depth h_0, such that we write

$$h = h_0 + \varepsilon h_1(x) + \ldots,$$

where ε is a small quantity expressing the magnitude of deviations from uniform. Similarly we also let the possible non-constant slope be

$$S = S_0 + \varepsilon S_1(x) + \ldots.$$

In a real stream varying along its length, both Ω and F are functions of x and h. We write the series:

$$\Omega = \Omega_0 + \varepsilon \Omega_1(x) + \varepsilon h_1(x) \Omega_0' + O\left(\varepsilon^2\right),$$

where Ω_1 is a change caused by a change in the channel properties, whether the resistance coefficient or the cross-section, and $\Omega_0' = \mathrm{d}\Omega/\mathrm{d}h|_0$. We also write

$$\mathsf{F}^2 = \mathsf{F}_0^2 + O\left(\varepsilon\right) + \ldots,$$

in which we will find that terms in ε are not necessary. Multiplying through by $1 - \beta \mathsf{F}^2$, setting dh_0/dx to zero for uniform flow and neglecting terms in ε^2:

$$\varepsilon \left(1 - \beta \mathsf{F}_0^2\right) \frac{\mathrm{d}h_1(x)}{\mathrm{d}x} = S_0 + \varepsilon S_1(x) - Q^2 \left(\Omega_0 + \varepsilon \Omega_1(x) + \varepsilon h_1(x) \Omega_0'\right)$$

At zeroeth order ε^0 we obtain

$$S_0 = Q^2 \Omega_0$$

an expression of whichever flow formula is being used, and is identically satisfied. At ε^1, we obtain the linear differential equation

$$\frac{\mathrm{d}h_1}{\mathrm{d}x} - \gamma h_1 = \phi(x)$$

where γ is a constant:

$$\gamma = -\frac{S_0 \Omega_0' / \Omega_0}{1 - \beta \mathsf{F}_0^2} = \frac{S_0}{1 - \beta \mathsf{F}_0^2} \times \begin{cases} 2 \dfrac{\mathrm{d}Q_\mathrm{r}/\mathrm{d}h|_0}{Q_{\mathrm{r}0}}, & \text{General expression,} \\ 3 \dfrac{B_0}{A_0} - \dfrac{\mathrm{d}P/\mathrm{d}h|_0}{P_0}, & \text{Chézy-Weisbach ,} \\ \frac{10}{3} \dfrac{B_0}{A_0} - \frac{4}{3} \dfrac{\mathrm{d}P/\mathrm{d}h|_0}{P_0}, & \text{Gauckler-Manning,} \end{cases}$$

and the forcing term on the right is

$$\phi(x) = \frac{S_0}{1 - \beta \mathsf{F}_0^2} \left(\frac{S_1(x)}{S_0} - \frac{\Omega_1(x)}{\Omega_0} \right), \tag{1}$$

showing the effects of fractional changes in slope and resistivity Ω.

Solving the differential equation

The differential equation is in *integrating factor* form, and can be solved by multiplying both sides by $e^{-\gamma x}$ and writing the result

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\gamma x}h_1\right) = e^{-\gamma x}\phi\left(x\right),$$

which can be integrated to give

$$h_1 = e^{\gamma x}\left(\int^{x} e^{-\gamma x'}\phi\left(x'\right)\,dx' + \text{Constant}\right),$$

where x' is a dummy variable. Returning to physical variables, $h = h_0 + \varepsilon h_1$ gives the solution

$$h = h_0 + He^{\gamma x} + \int^{x} e^{\gamma\left(x-x'\right)}\phi\left(x'\right)\,dx'$$

The part of the solution $He^{\gamma x}$ is that obtained by Samuels (1989), giving the solution for backwater level in a uniform channel by evaluating the constant of integration using a downstream boundary condition $h = H$ at $x = 0$. The part of the solution shows how the surface decays upstream at a rate $e^{\gamma x}$, as x becomes increasingly negative, because γ is positive,

- For a wide channel, the terms in dP/dh in the formulae for γ are unimportant (and are often not well known), so that $A_0/B_0 \approx h_0$, the channel depth, and for small Froude number this gives

$$\gamma \approx 3\frac{S_0}{h_0},$$

showing that the rate of exponential decay is small for gently sloping and deep streams and greatest for steep and shallow ones.

- Consider the distance $x_{1/2}$ upstream for the effect of a downstream surface elevation to diminish by a factor of 1/2. Then $\exp\left(-\gamma x_{1/2}\right) = 1/2$, or

$$x_{1/2} = \frac{\ln 2}{\gamma} \approx \frac{\ln 2}{3}\frac{h_0}{S_0} \approx 0.2\frac{h_0}{S_0}$$

So for a gently-sloping river $S_0 = 10^{-4}$ and 2 m deep, the effect of any backwater decreases by 1/2 in a distance of 4 km. To diminish to 1/16, say, the distance is 16 km. For a steeper river, say $S_0 = 0.0016$ from the example simulated above, where $h_0 \approx 1$ m, the "half-length" is about 150 m. This is roughly in agreement with the computed results above.

- If the approximate exponential decay solution were shown on that figure, it would not agree closely with the computed results, because the checked-up disturbance is as large as 50% of the depth, when the linear solution is not all that accurate. The beauty of Samuels' result is in its ability to give a quick estimate and an appreciation of the quantities that affect the length of backwater.

General solution for channel

Here we neglect any boundary conditions and consider just the solution due to the forcing function ϕ due to changes in the channel:

$$h = h_0 - \int_x^{\infty} e^{\gamma\left(x-x'\right)} \phi\left(x'\right)\, dx'$$

- The integral expresses the effect of all downstream channel variations, expressed as a convolution integral of the disturbance function ϕ and the exponential decay function with a length scale at the same γ.
- At the general point x in subcritical flow, the disturbance is due to the integrated effects of the disturbance function ϕ for all downstream points, from x to ∞, weighted according to the exponential decay function.

Example: The effect on a river of a finite length of greater resistance

Consider, as an example, a case where over a finite length L of river, the carrying capacity is reduced by the resistivity Ω increasing by a relative amount $\Omega_1/\Omega_0 = \delta$, such as by local deposition of material, between $x = 0$ and $x = L$, and constant in that interval. Assume F_0^2 negligible and the river wide. The forcing function is:

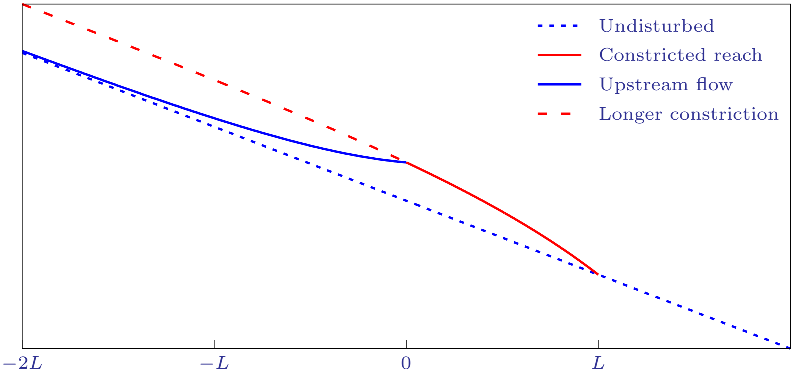
$$\phi(x) = \begin{cases} 0, & \text{if } x \leqslant 0; \\ -S_0\delta, & \text{if } 0 \leqslant x \leqslant L; \\ 0, & \text{if } x \geqslant L. \end{cases}$$

- For x downstream, $x \geqslant L$, $\phi(x) = 0$, and $h = h_0$, which is correct in this sub-critical flow, there are no downstream effects.
- For x in the section where the changes occur, $0 \leqslant x \leqslant L$, the solution is

$$h = h_0 + S_0\delta \int_x^L e^{\gamma\left(x-x'\right)}\, dx' = h_0 + \frac{S_0\delta}{\gamma}\left(1 - e^{\gamma(x-L)}\right).$$

- For x upstream, $x \leqslant 0$, where there is no extra resistance,

$$h = h_0 + S_0\delta e^{\gamma x} \int_0^L e^{-\gamma x'}\, dx' = h_0 + \frac{S_0\delta}{\gamma} e^{\gamma x}\left(1 - e^{-\gamma L}\right).$$



These solutions are all shown in the figure with an arbitrary vertical scale such that the slope is exaggerated. The calculations were performed for $S_0 = 0.0005$, $h_0 = 1\,\mathrm{m}$, and with a constricted length of $L = 1000\,\mathrm{m}$, with a 10% increase in resistance there, such that $\delta = 0.1$. Using these figures, and with $\gamma = 3S_0/h_0$, the computed backwater at the beginning of the constriction calculated according to the formula was 2.6 cm. In the reach of increased resistance the surface is raised, as one expects and shows an exponential approach to the changed depth $S_0\delta/\gamma$ if $L \to \infty$.

- The abrupt changes of gradient violate our physical assumptions of the long wave equations, but they give us a clear picture of what happens, possibly obvious in retrospect, but hopefully of assistance.
- We have made an approximate model, with very little input data necessary, and we have correspondingly approximate results.

Data-based models with transfer functions – linear and maybe nonlinear

Data-based model with transfer functions

- In many problems little is known about the nature of a stream or streams, their geometry or resistance.
- Knowledge of input and output time series with linear convolutions can be used with to model even a complicated river system.
- There have been innumerable papers on the subject of the unit hydrograph and linear and nonlinear systems in general, including some sophisticated additions such as Artificial Neural Networks.
- The ability to identify the system using optimising software is rather simpler than with other deconvolution methods.
- Input sequence of I_m, $m = 0, \ldots, M-1$, which could be river levels or flows
- Linear transfer function h_k, $k = 0, \ldots, K$, relating input to output
- Output O_n, whether river level or flow, is due to all the contributions I_m multiplied by their effect on the outflow with a time difference $n - m$:

$$O_n = \sum_{\substack{m=0, \\ m>n-K}}^{n \leqslant M} I_m h_{n-m}, \quad \text{for } n = 0, \ldots, N-1.$$

- Such a summation is a discrete convolution. First, one takes the M input values and N output values and solves the system of linear equations for the h_k by standard methods. Then, the effects of any future flood can be predicted by performing the convolutions with the calculated h_k but with a new set of observed I_m.

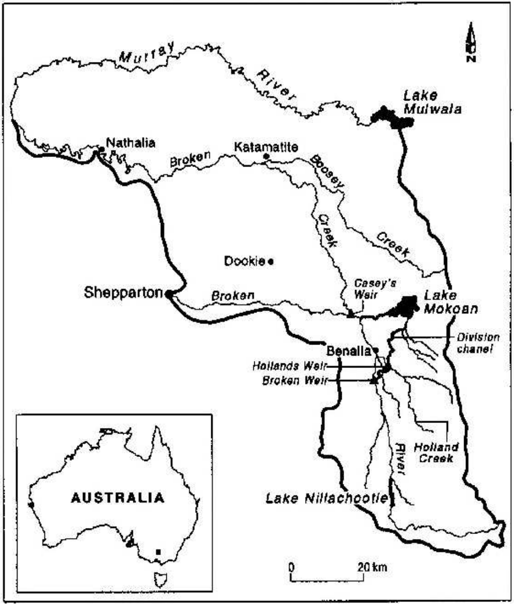
Using optimisation

- There may be problems in solving for the transfer function.
- The system of equations might be over-determined and might be poorly conditioned numerically.
- The use of optimising software overcomes some of these problems – *and would even allow nonlinear generalisations*. We seek to minimise the total sum of the squares of the errors e of the approximating convolutions over the N data points:

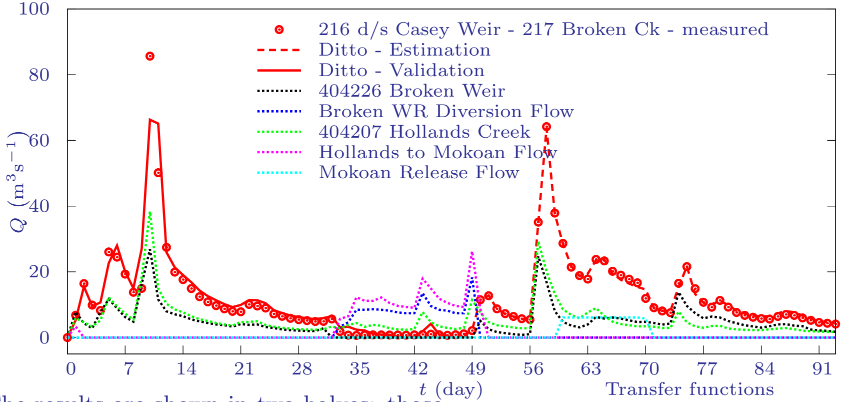
$$e = \sum_{n=0}^{N-1} \left(\sum_{\substack{m=0,\\ m>n-K}}^{n \leqslant M} I_m h_{n-m} - O_n \right)^2 .$$

- Such a method was used in a study of flows in a complex set of interconnections in the Broken River Valley in south-eastern Australia
- The routing model was expressed as the simple combination of several transfer functions such as that shown previously.

The Broken River Valley



Results



The results are shown in two halves: those for $t > 45\,\mathrm{d}$ were used to estimate the transfer functions (there was a block release flow from Lake Mokoan between $59 - 71\,\mathrm{d}$ whereas there was no release in the other half of the data, for $t < 45\,\mathrm{d}$ so that it would not have been possible to determine the transfer function for it) .