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Numerical analysis of steady gradually varied flow in open channel networks with hydraulic structures

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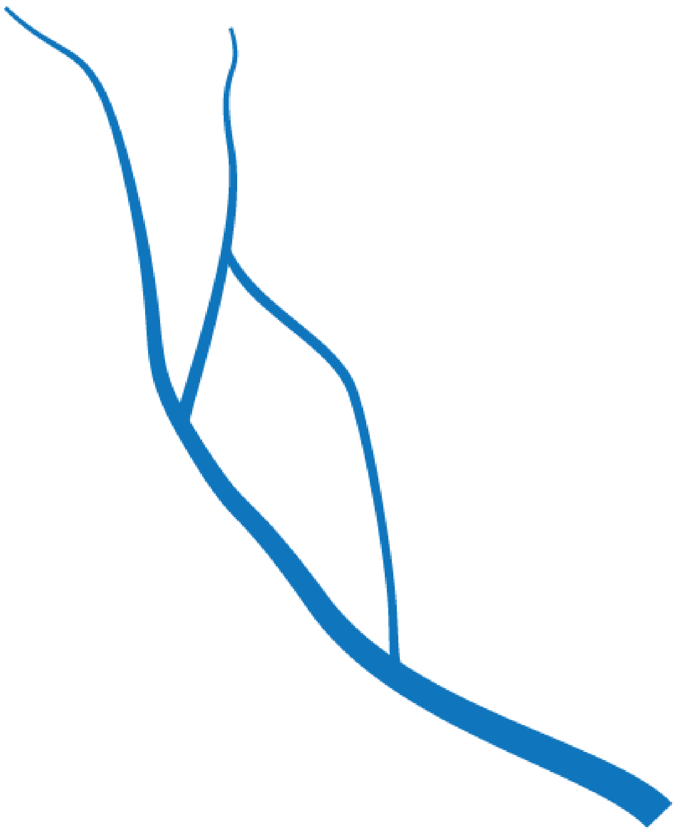
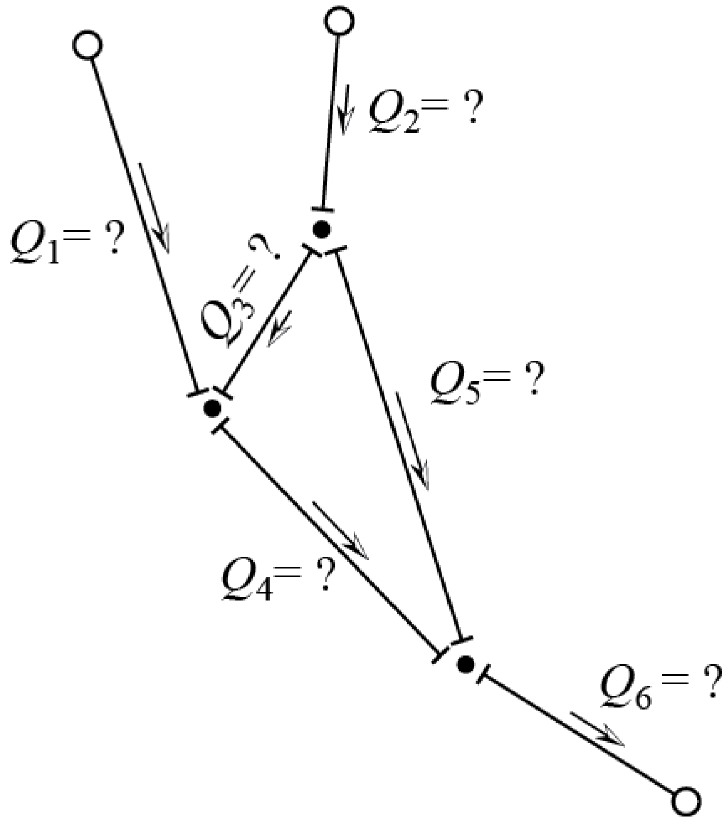
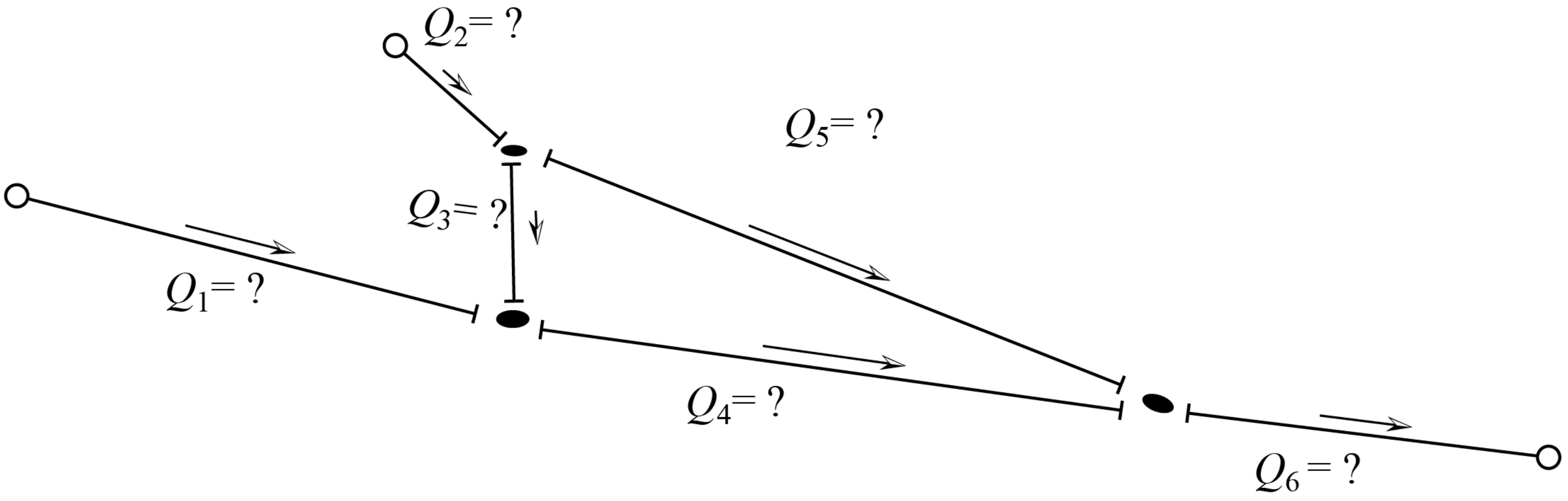
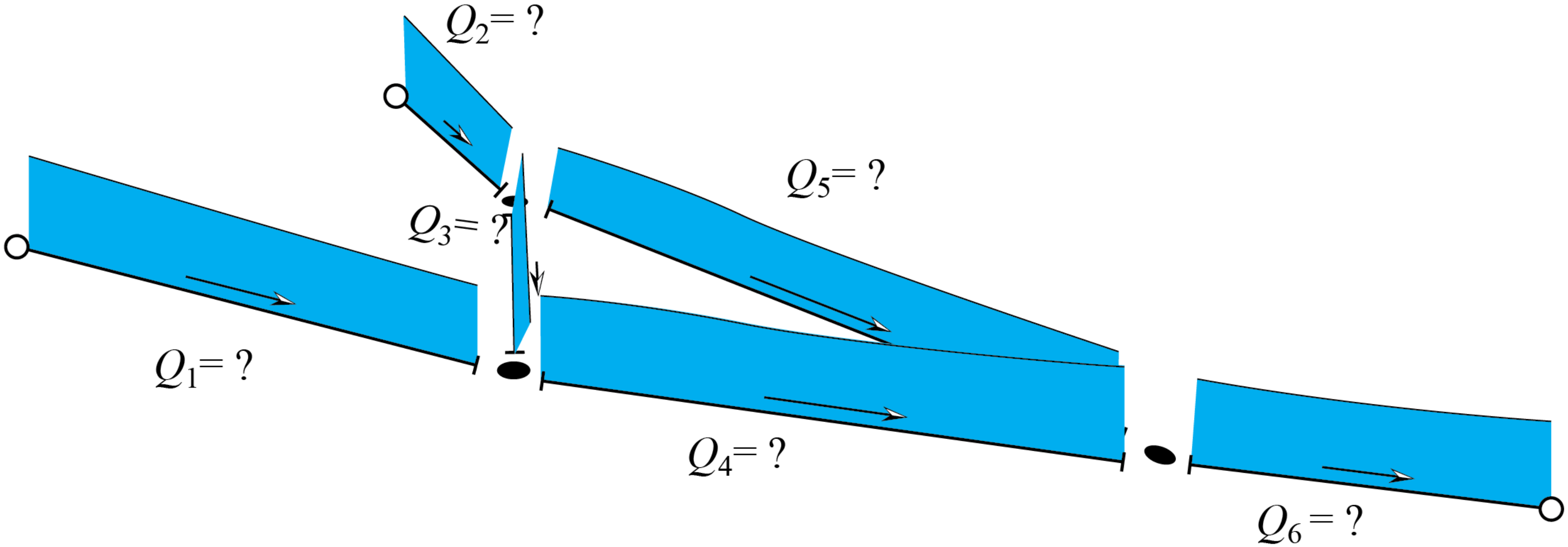
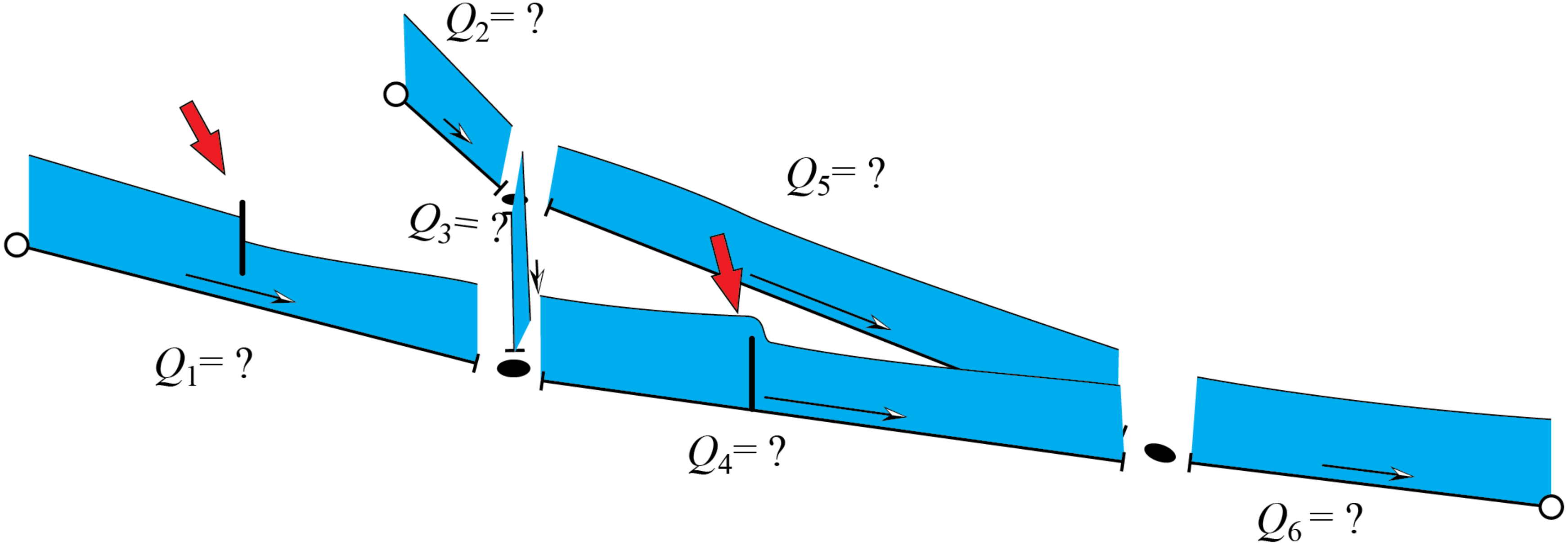



Q_1= ?
Q_2= ?
Q_3= ?
Q_4= ?
Q_5= ?
Q_6 = ?


Q2= ?
Q5= ?
Q3= ?
Q1= ?
Q4= ?
Q6 = ?


$Q_2 = ?$
$Q_5 = ?$
$Q_3 = ?$
$Q_1 = ?$
$Q_4 = ?$
$Q_6 = ?$


Q_2= ?
Q_5= ?
Q_3= ?
Q_1= ?
Q_4= ?
Q_6 = ?

Steady gradually varied flow governing equation

$$\frac{dE}{dx} = -S$$

where

x – spatial coordinate [L]

E – height of the mechanical energy of the flow [L]

S – the longitudinal slope of the mechanical energy [-]

Mechanical energy

$$E = h + \frac{\alpha \cdot Q^2}{2g \cdot A^2}$$

Energy slope

$$S = \frac{n^2 \cdot Q^2}{A^2 \cdot R^{4/3}}$$

h – water stage level [L]

α – energy correctional coefficient [-]

Q – flow discharge $[L^3/T]$

n – Manning's roughness coefficient $[T/L^{1/3}]$

g – gravitational acceleration $[L^2/T]$

A – active flow area $[L^2]$

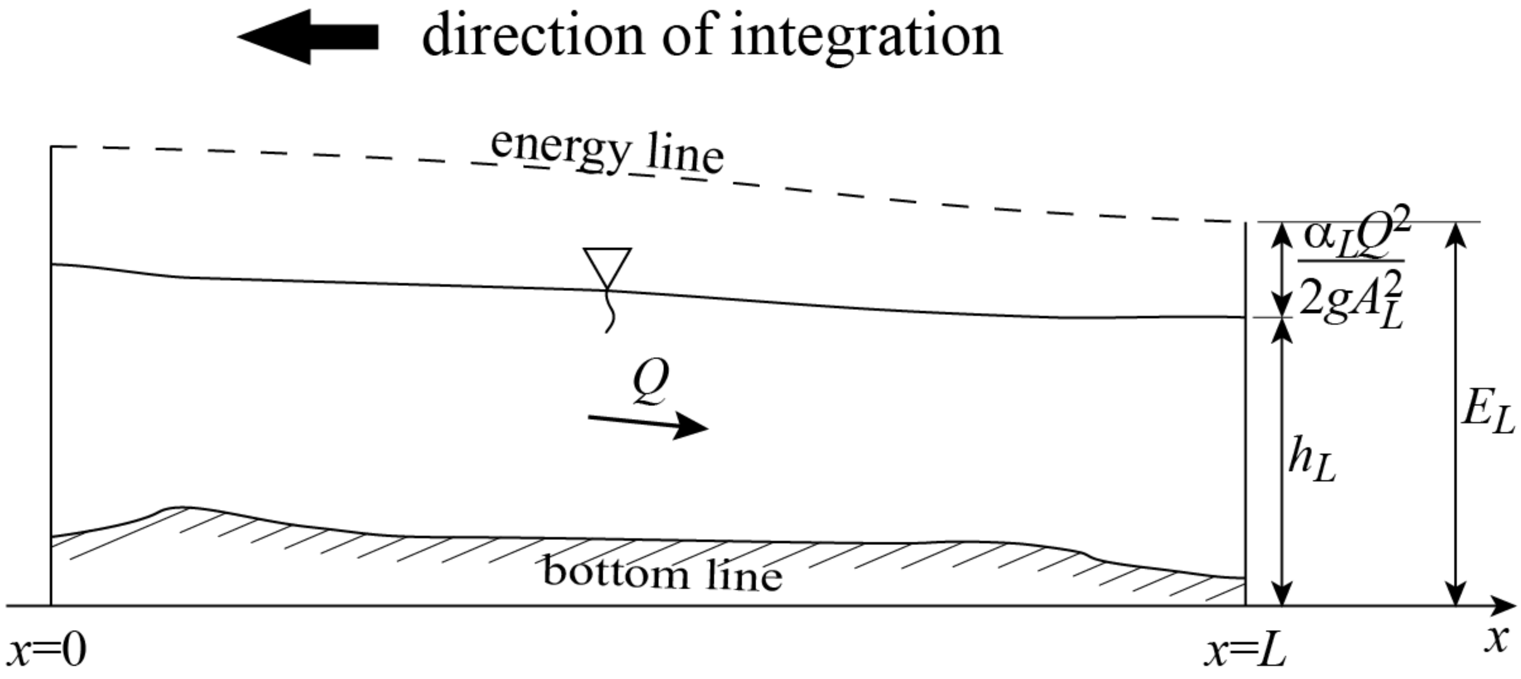
R – hydraulic radius [L]

Steady gradually varied flow governing equation

$$\frac{d}{dx}\left(h+\frac{\alpha \cdot Q^2}{2g \cdot A^2}\right)=-\frac{n^2 \cdot Q^2}{A^2 \cdot R^{4/3}}$$

Initial value problem for ODE

$$\frac{d}{dx}\left(h+\frac{\alpha\cdot Q^2}{2g\cdot A^2}\right)=-\frac{Q^2\cdot n^2}{A^2\cdot R^{4/3}} \quad \text{with} \quad E(x=L)=E_L$$



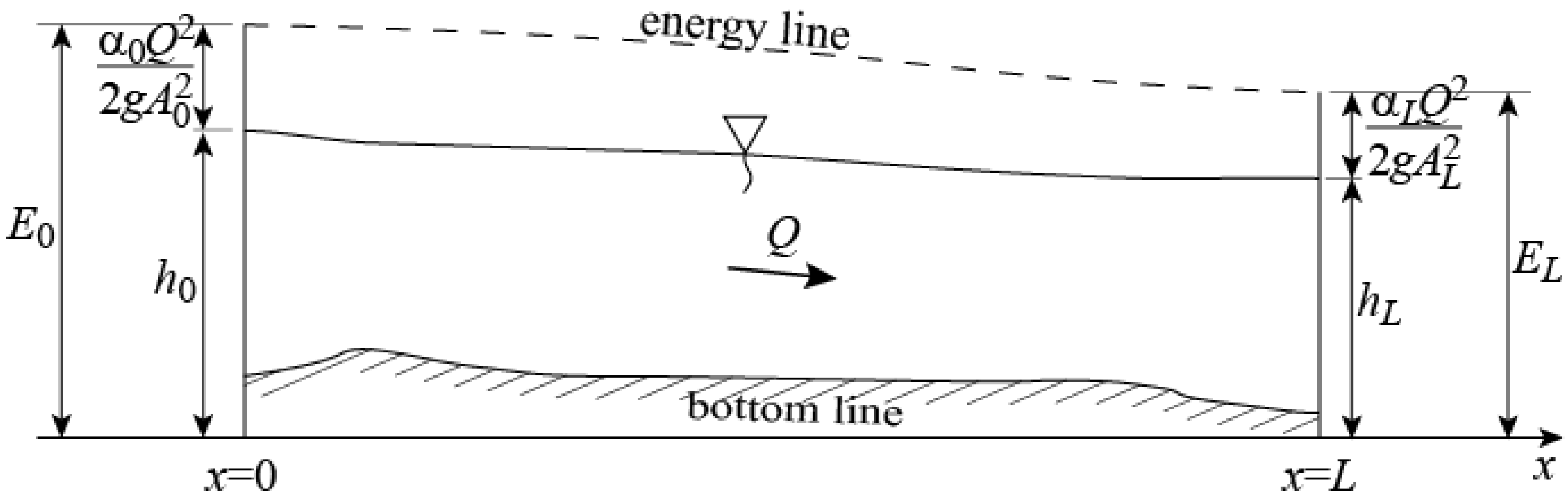
Boundary problem for energy equation

$$\begin{cases} \dfrac{dQ}{dx} = 0 \\ \dfrac{dE}{dx} = -S \end{cases}$$

with

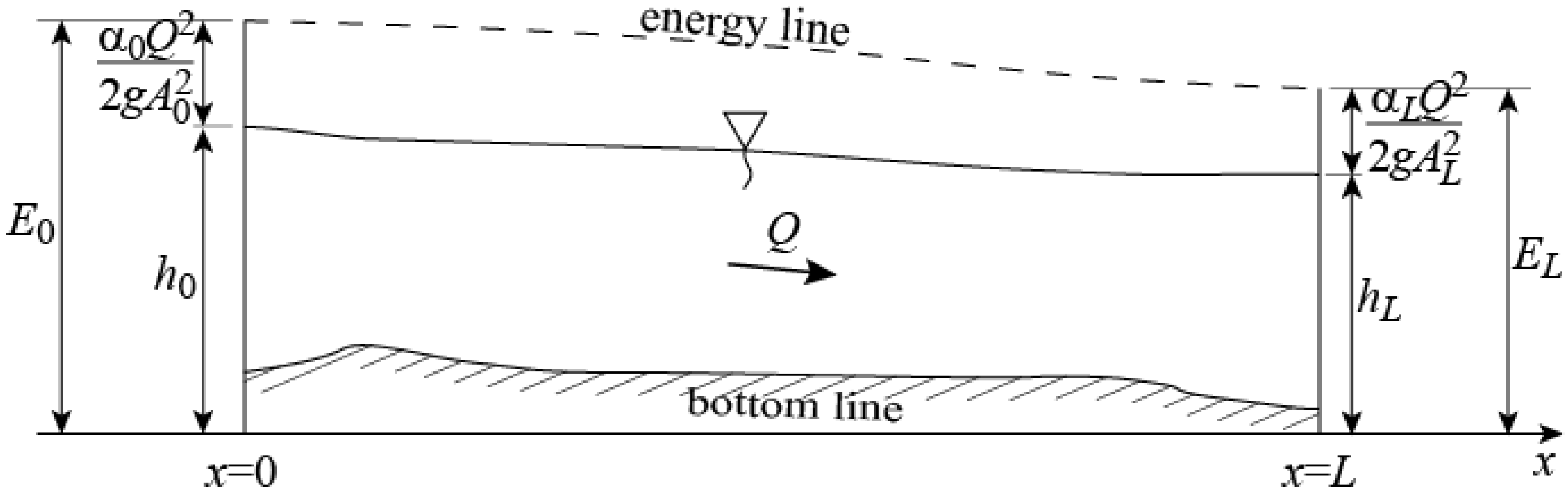
$$E(x=0) = E_0$$

$$E(x=L) = E_L$$



Boundary problem for energy equation

$$\begin{cases} Q = const. \\ \dfrac{dE}{dx} = -S \end{cases} \quad \text{with} \quad \begin{aligned} E(x=0) &= E_0 \\ E(x=L) &= E_L \end{aligned}$$



Solution of the boundary problem for ODE

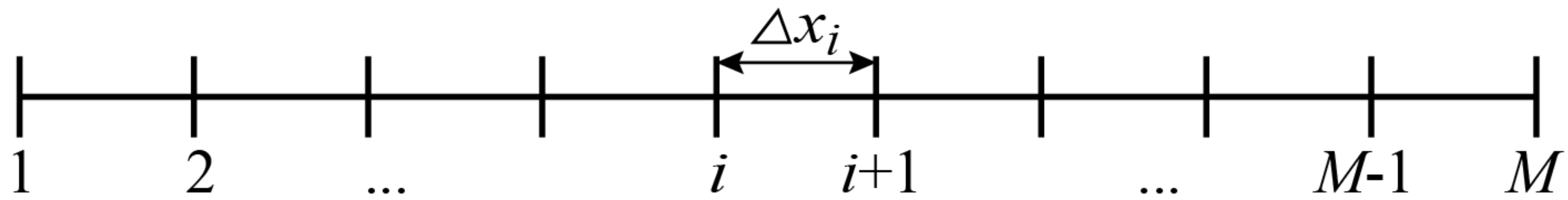
- the shooting method
- the finite difference method

Solution of the boundary problem for ODE

- **the shooting method** - repetitive solution of initial value problems

- **the finite difference method** – unified and generalized problem formulation and its solution

Discretization of the channel



Discrete approximation of the energy equation

$$E_{i+1} - E_i + \frac{\Delta x_i}{2}\left(S_i + S_{i+1}\right) = 0$$

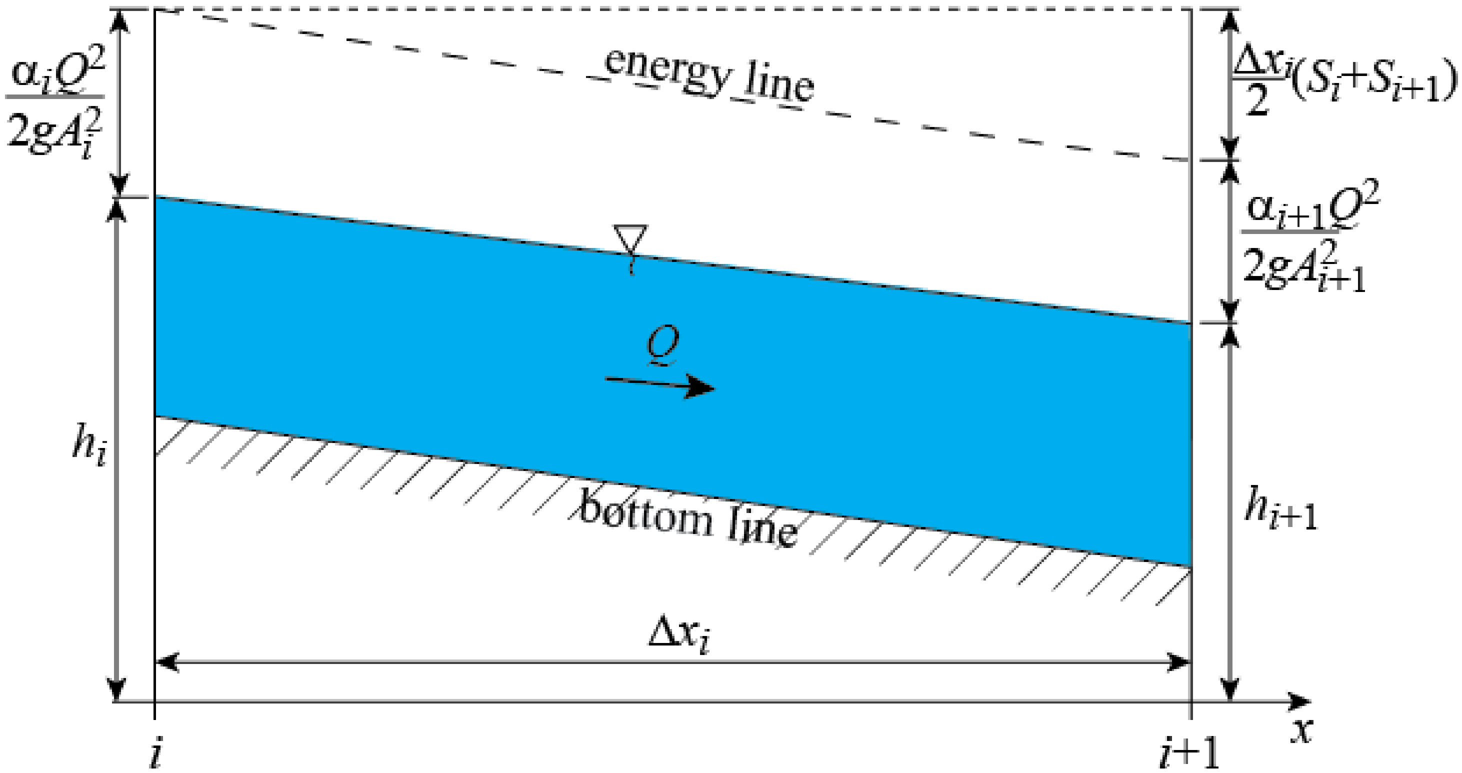
$$i = 1, \ldots, M-1$$

Δx_i - distance between i^{th} and $(i+1)^{th}$ cross-section

i – computational cross-section index

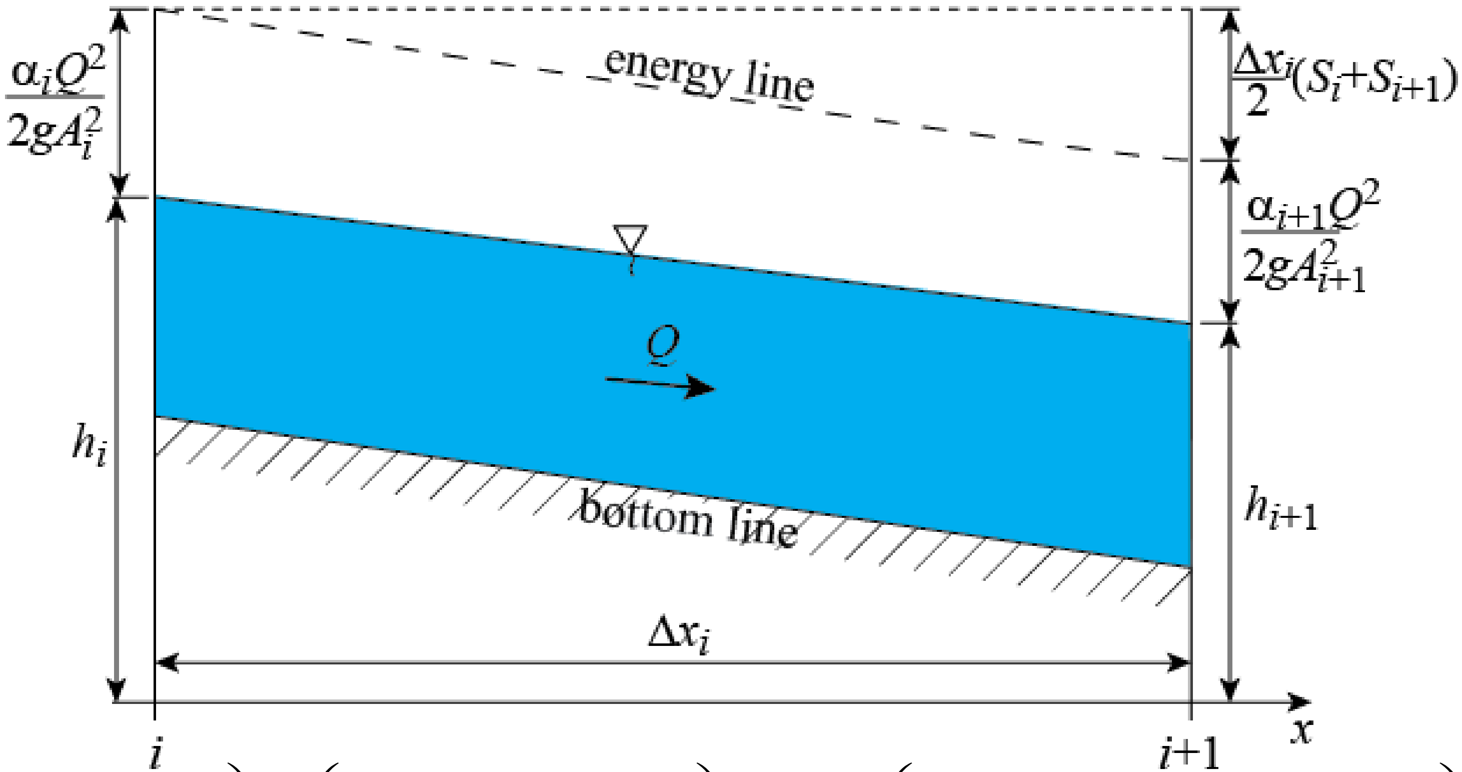
M – number f computational cross-sections

Discrete form of energy equation interpretation



$$E_{i+1} - E_i + \frac{\Delta x_i}{2}\left(S_i + S_{i+1}\right) = 0$$

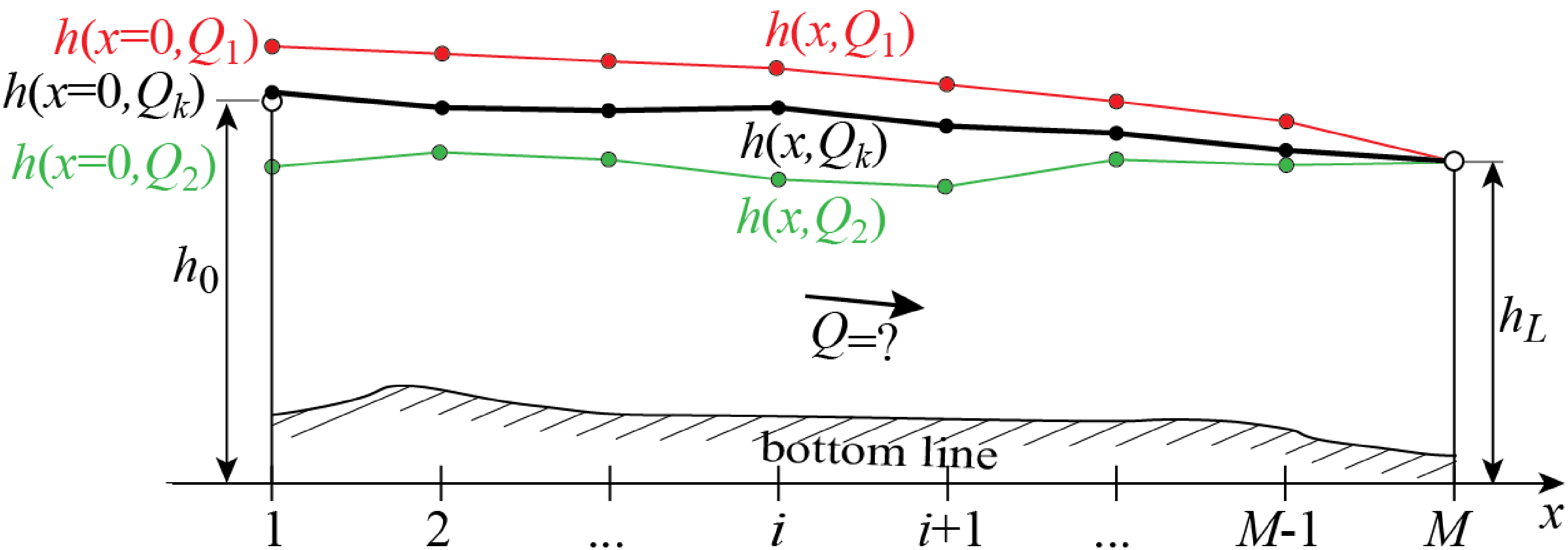
Discrete form of energy equation interpretation



$$-\left(h_i + \frac{\alpha_i \cdot Q^2}{2g \cdot A_i^2}\right) + \left(h_{i+1} + \frac{\alpha_{i+1} \cdot Q^2}{2g \cdot A_{i+1}^2}\right) + \frac{\Delta x_i}{2}\left(\frac{Q^2 \cdot n_i^2}{A_i^2 \cdot R_i^{4/3}} + \frac{Q^2 \cdot n_{i+1}^2}{A_{i+1}^2 \cdot R_{i+1}^{4/3}}\right) = 0$$

Shooting method approach

Solution of the set of initial value problems for the energy equation



Finite difference method approach

Formulation and solution of the system of equations

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

Unknowns

- water stage levels at each cross-section

$$h_i \qquad (i = 1, \ldots, M)$$

- flow discharge

$$Q$$

Including the flow direction

$$-\left(h_i+\frac{\alpha_i\cdot Q^2}{2g\cdot A_i^2}\right)+\left(h_{i+1}+\frac{\alpha_{i+1}\cdot Q^2}{2g\cdot A_{i+1}^2}\right)+\frac{\Delta x_i}{2}\left(\frac{Q^2\cdot n_i^2}{A_i^2\cdot R_i^{4/3}}+\frac{Q^2\cdot n_{i+1}^2}{A_{i+1}^2\cdot R_{i+1}^{4/3}}\right)=0$$

Including the flow direction

$$-\left(h_i + \frac{\alpha_i \cdot Q^2}{2g \cdot A_i^2}\right) + \left(h_{i+1} + \frac{\alpha_{i+1} \cdot Q^2}{2g \cdot A_{i+1}^2}\right) + \frac{\Delta x_i}{2}\left(\frac{Q^2 \cdot n_i^2}{A_i^2 \cdot R_i^{4/3}} + \frac{Q^2 \cdot n_{i+1}^2}{A_{i+1}^2 \cdot R_{i+1}^{4/3}}\right) = 0$$

$$Q^2 \rightarrow Q \cdot |Q|$$

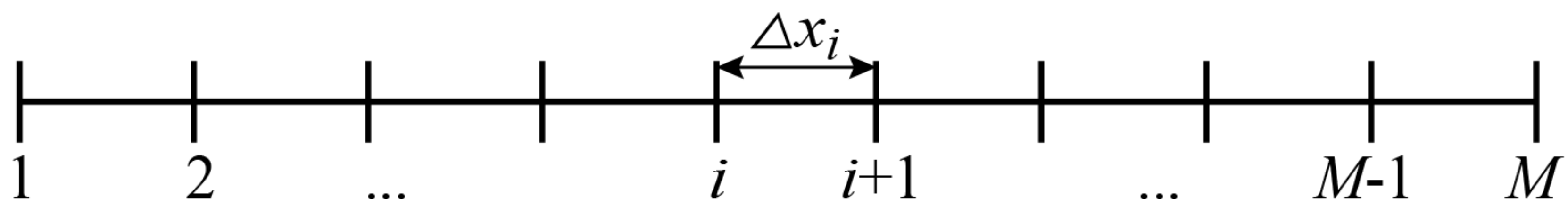
Including the flow direction

$$-\left(h_i+\frac{\alpha_i\cdot Q^2}{2g\cdot A_i^2}\right)+\left(h_{i+1}+\frac{\alpha_{i+1}\cdot Q^2}{2g\cdot A_{i+1}^2}\right)+\frac{\Delta x_i}{2}\left(\frac{Q^2\cdot n_i^2}{A_i^2\cdot R_i^{4/3}}+\frac{Q^2\cdot n_{i+1}^2}{A_{i+1}^2\cdot R_{i+1}^{4/3}}\right)=0$$

$$Q^2\to Q\cdot|Q|$$

$$-\left(h_i+\frac{\alpha_i\cdot Q^2}{2g\cdot A_i^2}\right)+\left(h_{i+1}+\frac{\alpha_{i+1}\cdot Q^2}{2g\cdot A_{i+1}^2}\right)+\frac{\Delta x_i}{2}\left(\frac{Q\cdot|Q|\cdot n_i^2}{A_i^2\cdot R_i^{4/3}}+\frac{Q\cdot|Q|\cdot n_{i+1}^2}{A_{i+1}^2\cdot R_{i+1}^{4/3}}\right)=0$$

System of non-linear algebraic equations



$$\begin{cases} h_1 = h_0 \\ -\left(h_i + \dfrac{\alpha_i \cdot Q^2}{2g \cdot A_i^2} \right) + \left(h_{i+1} + \dfrac{\alpha_{i+1} \cdot Q^2}{2g \cdot A_{i+1}^2} \right) + \dfrac{\Delta x_i}{2} \left(\dfrac{Q \cdot |Q| \cdot n_i^2}{A_i^2 \cdot R_i^{4/3}} + \dfrac{Q \cdot |Q| \cdot n_{i+1}^2}{A_{i+1}^2 \cdot R_{i+1}^{4/3}} \right) = 0 \\ h_M = h_L \qquad\qquad i = 1, \ldots, M-1 \end{cases}$$

Structure of the arising system of non-linear equations

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} a_{1,1} & & & & & & & \\ a_{2,1} & a_{2,2} & & & & & & a_{2,M+1} \\ & \ddots & \ddots & & & & & \vdots \\ & & a_{i,i-1} & a_{i,i} & & & & a_{i,M+1} \\ & & & \ddots & \ddots & & & \vdots \\ & & & & a_{M-1,M} & a_{M,M} & & a_{M,M+1} \\ & & & & & a_{M+1,M} & & \end{bmatrix} \begin{Bmatrix} h_1 \\ \vdots \\ h_i \\ h_{i+1} \\ \vdots \\ h_M \\ Q \end{Bmatrix} = \begin{Bmatrix} h_0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ h_L \end{Bmatrix}$$

Solution of non-linear system of algebraic equations

Modified Picard's method

$$\mathbf{A}^{*} \cdot \mathbf{x}^{(k+1)} = \mathbf{b}$$

$$\mathbf{A}^{*} = \mathbf{A}\left(\frac{\mathbf{x}^{(k-1)} + \mathbf{x}^{(k)}}{2} \right)$$

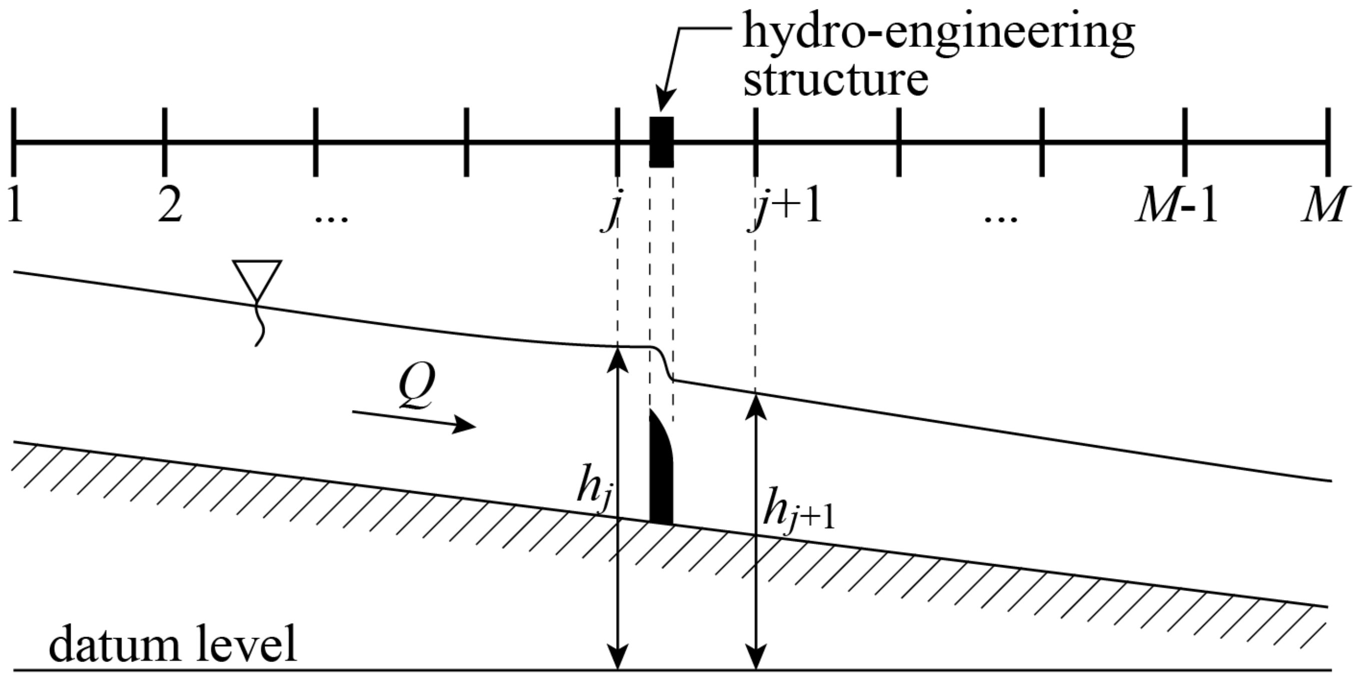
End of the iterative procedure

$$| h_i^{(k+1)} - h_i^{(k)} | < \varepsilon_h \qquad i = 1, \ldots, M-1$$

$$| Q^{(k+1)} - Q^{(k)} | < \varepsilon_Q$$

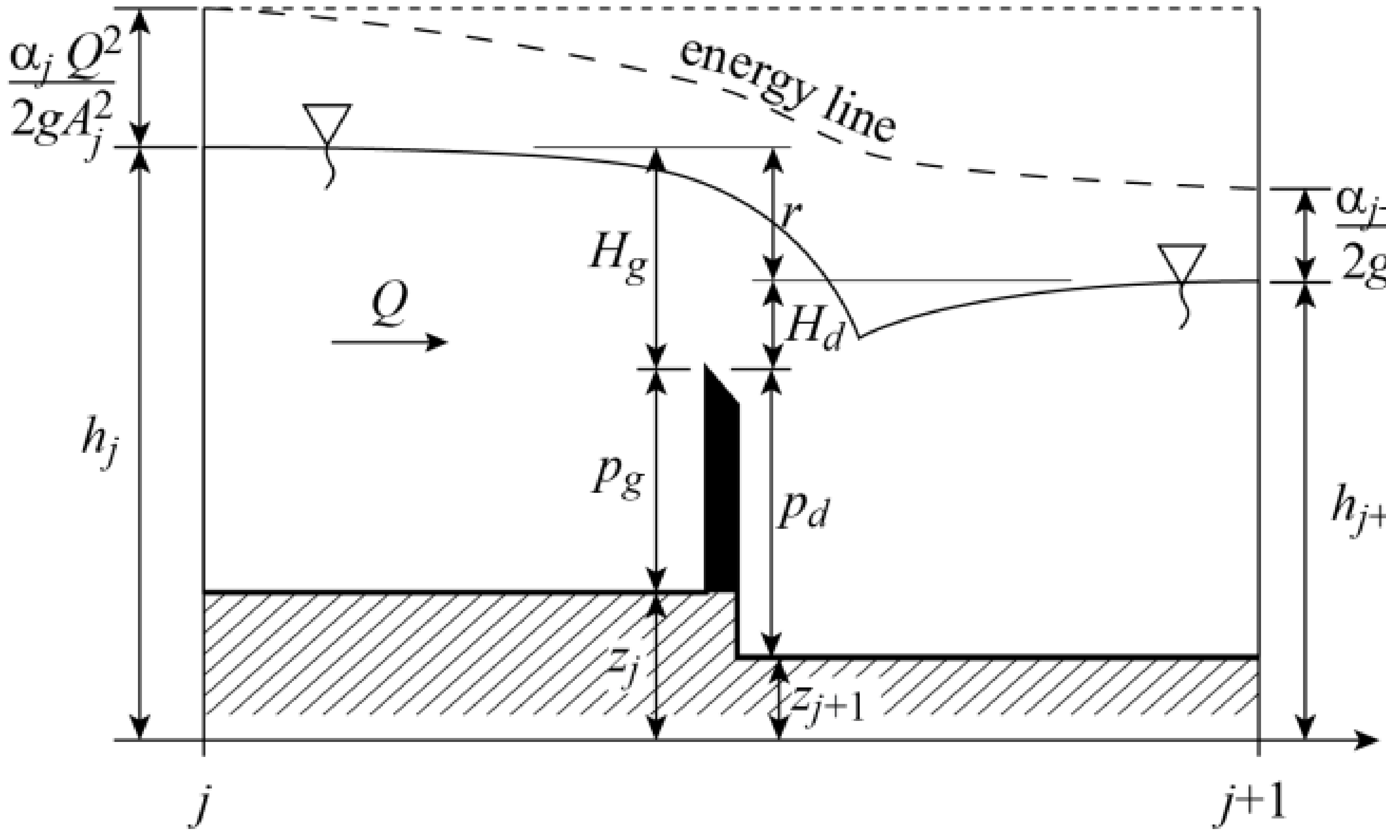
ε_h – water stage solution accuracy
ε_Q – flow discharge solution accuracy
k – iteration index

Including a hydraulic structure



$$Q = Q(h_j, h_{j+1})$$

An example: weir



Rectangular weir discharge formula

$$Q = \frac{2}{3}\mu \cdot B \cdot \sqrt{2g}\left[\left(H_g + \frac{\alpha_j \cdot Q^2}{2g \cdot A_j^2}\right)^{3/2} - \left(\frac{\alpha_j \cdot Q^2}{2g \cdot A_j^2}\right)^{3/2}\right] \cdot \sigma$$

$$\sigma = 1.05 \cdot \left(1 + 0.02\frac{H_d}{p_d}\right)\sqrt[3]{\frac{H_g - H_d}{H_g}}$$

μ – weir discharge coefficient [-]

σ – submersion coefficient [-]

B – weir width [L]

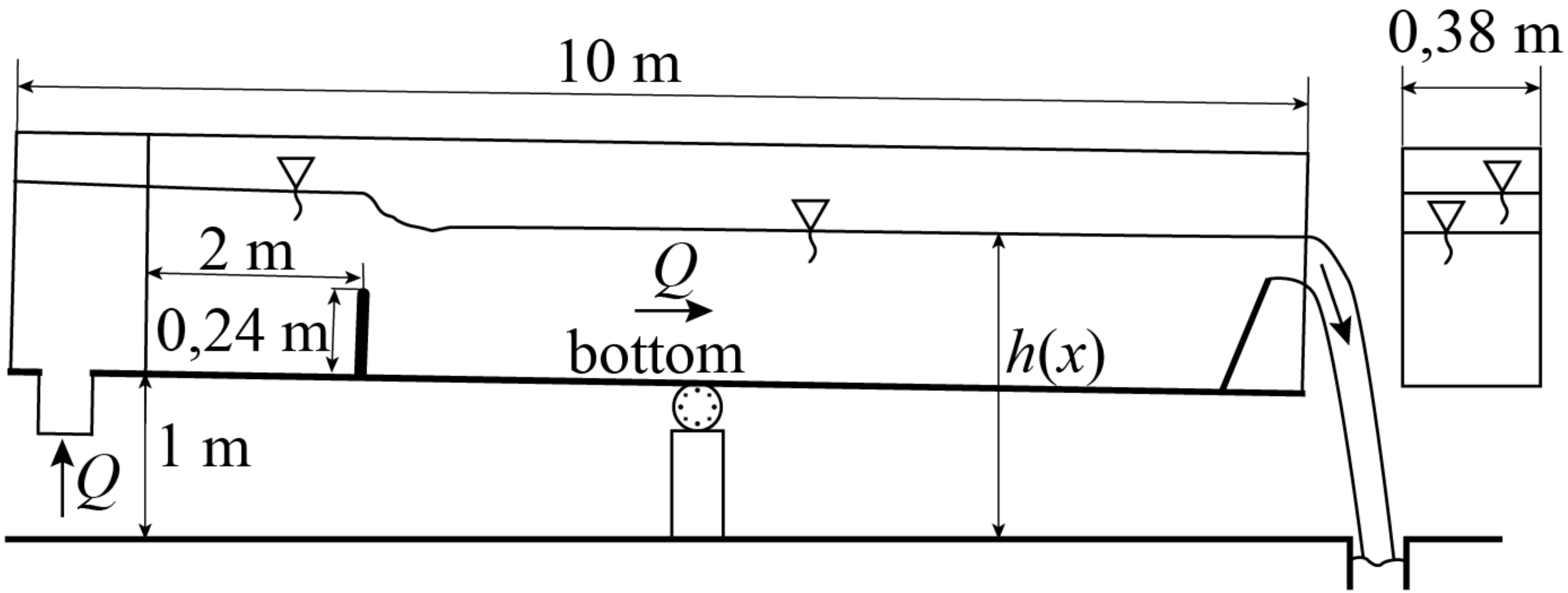
H_g – water level above the crest before the weir [L]

H_d – water level above the crest behind the weir [L]

p_g – crest level over the bottom before the weir [L]

p_d – crest level over the bottom behind the weir [L]

Physical experiment



$s = 0.001745$

$Q = 0.0133\,\mathrm{m}^3/\mathrm{s}$

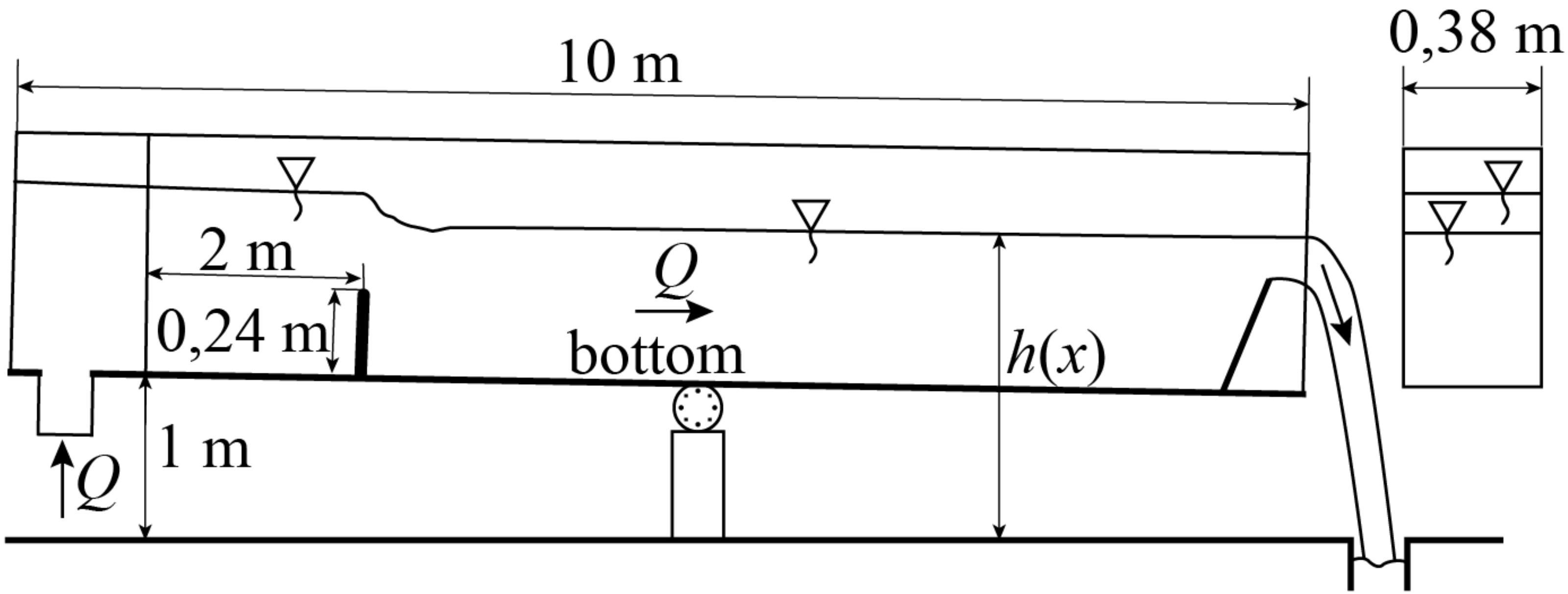
$n = 0.0185\,\mathrm{s/m}^{1/3}$

$h_0 = 1.31\,\mathrm{m}$

$p_g = p_d = 0.24\,\mathrm{m}$

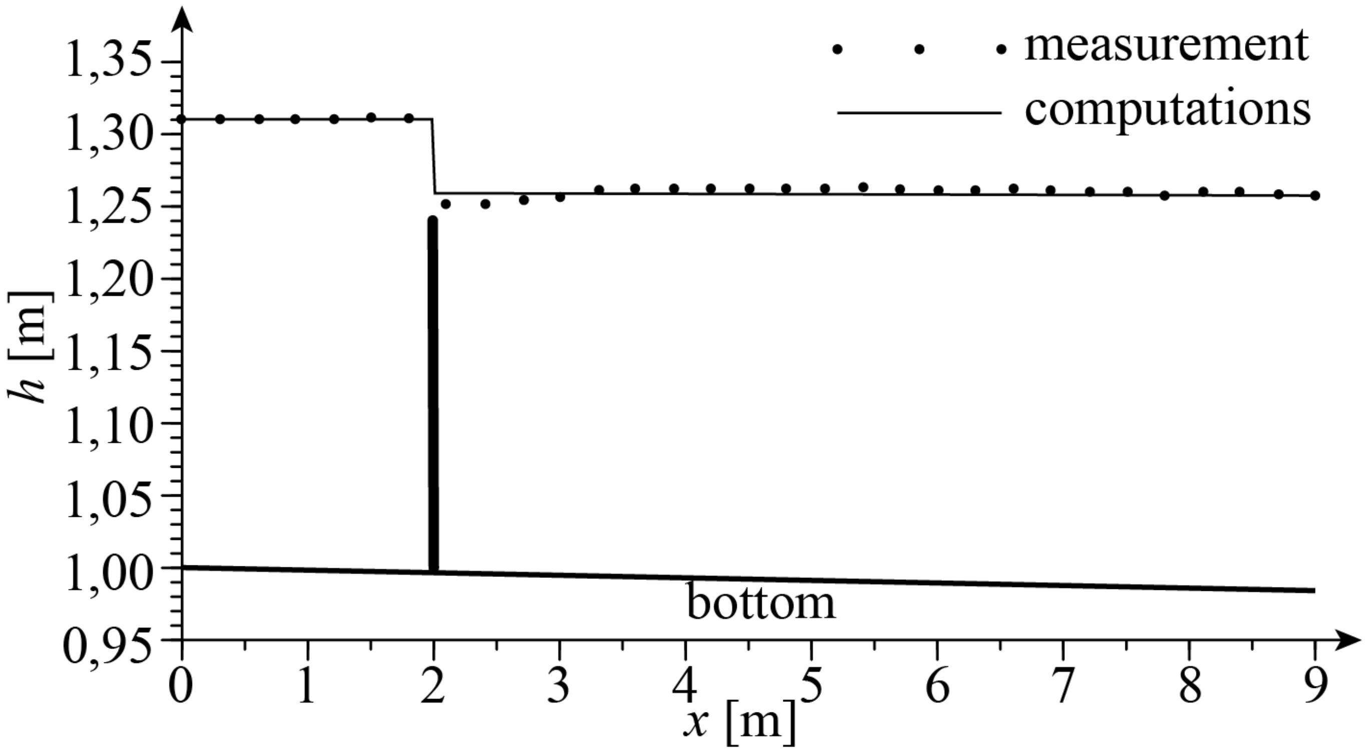
$h_L = 1.26\,\mathrm{m}$

Physical experiment



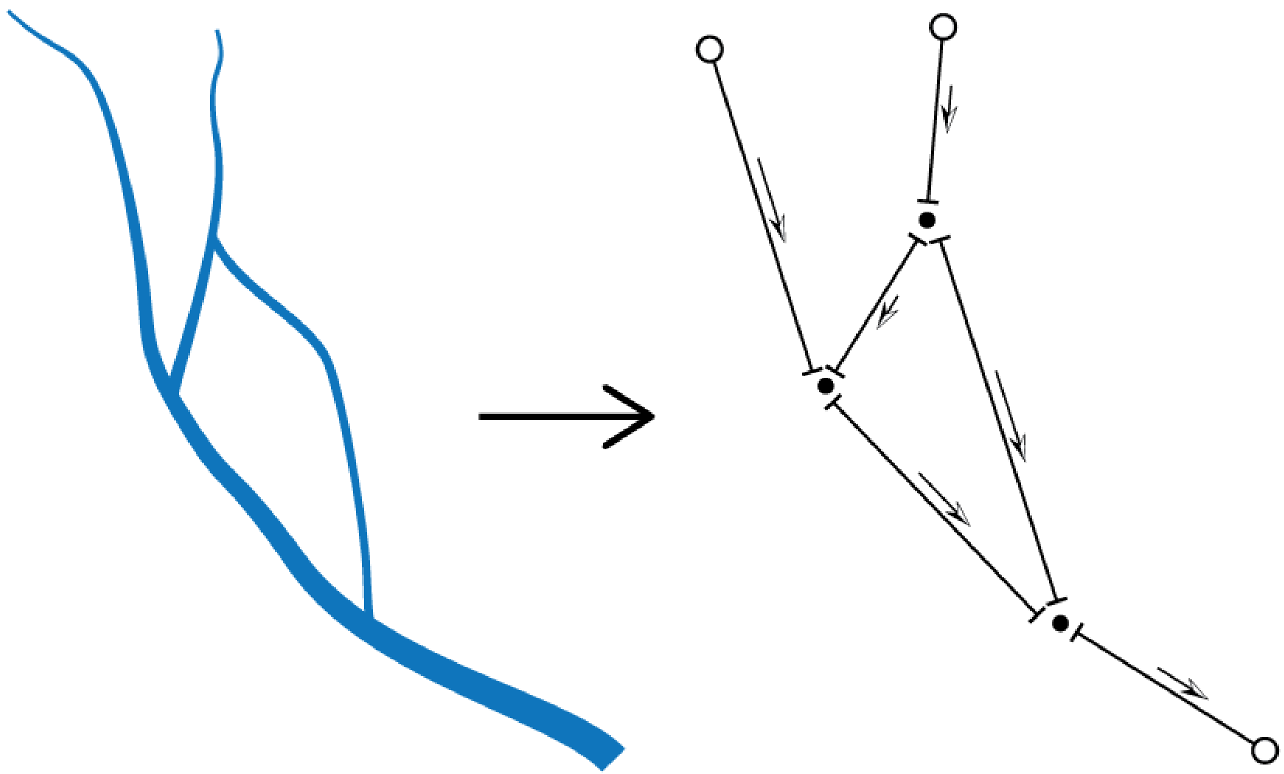
$$\mu_{ovf} = 0.615 \cdot \left(1 + \frac{1}{1000 \cdot H_g + 1.6}\right)\left[1 + 0.5\left(\frac{H_g}{H_g + p_g}\right)^2\right]$$

Measurement vs. computation output

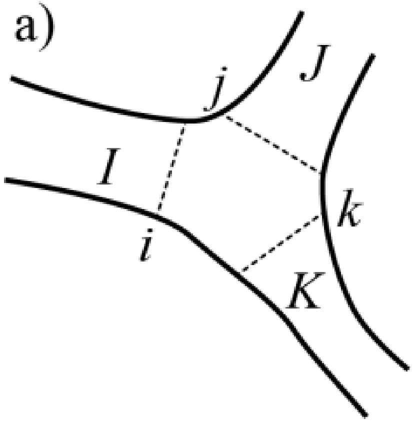


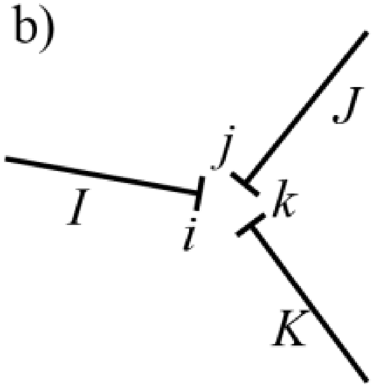
$$Q_{measured} = 0.0133 \frac{m^3}{s} \qquad Q_{computed} = 0.0142 \frac{m^3}{s} \qquad \varepsilon = 6.76\%$$

Channel networks



Channel network junctions

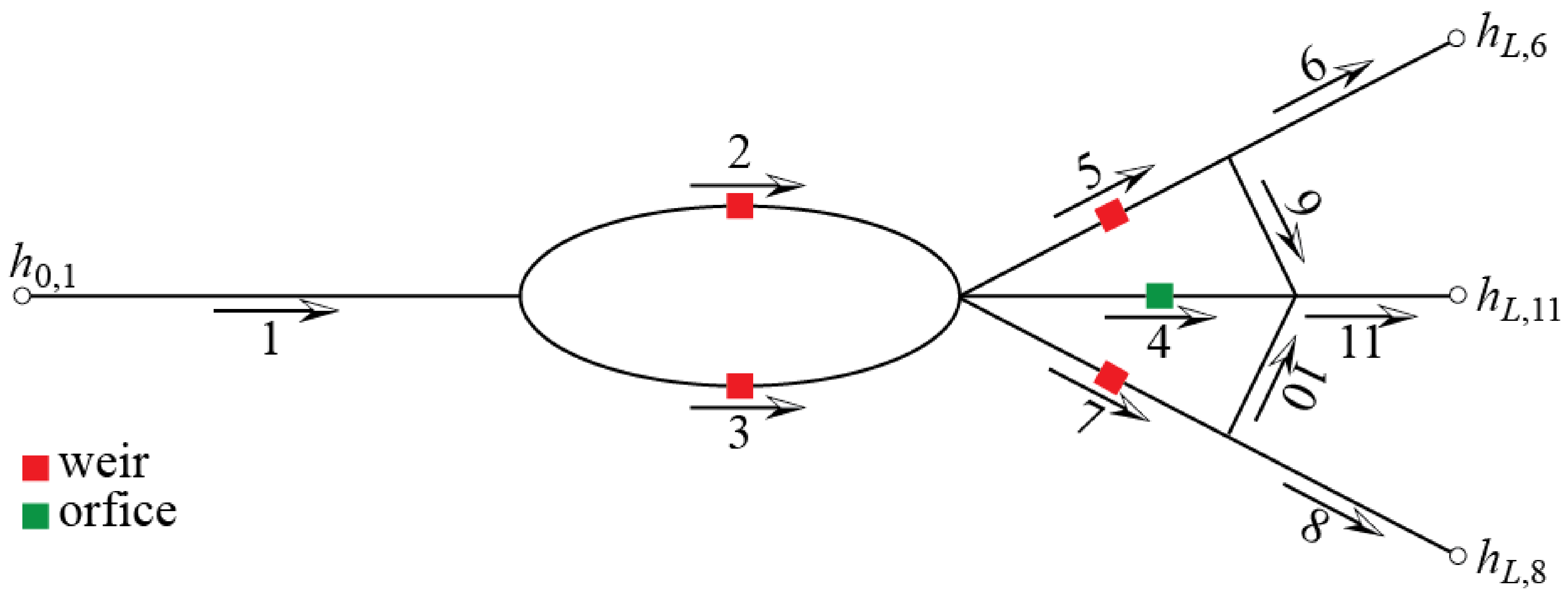




$$\sum_P Q_P = 0$$

$$h_i + \frac{\alpha_i \cdot Q_I^2}{2g \cdot A_i^2} = h_j + \frac{\alpha_j \cdot Q_J^2}{2g \cdot A_j^2} = h_k + \frac{\alpha_k \cdot Q_K^2}{2g \cdot A_k^2}$$

Numerical experiment



Channel network properties

No.	*s* [-]	*L* [m]
1	0.0001	2000
2	0.0001	1000
3	0.0001	1000
4	0.0001	1500
5	0.0001333	1500
6	0.0001	1500
7	0.0001333	1500
8	0.0001	1500
9	- 0.00005	1000
10	- 0.00005	1000
11	0.0002	1000

$$B_{ch} = 5\ \mathrm{m}$$

$$n_{ch} = 0.03\ \mathrm{s/m^{1/3}}$$

weir properties

$$B_{wr} = 5\ \mathrm{m}$$

$$p_{g,wr} = p_{d,wr} = 1\ \mathrm{m}$$

orfice properties

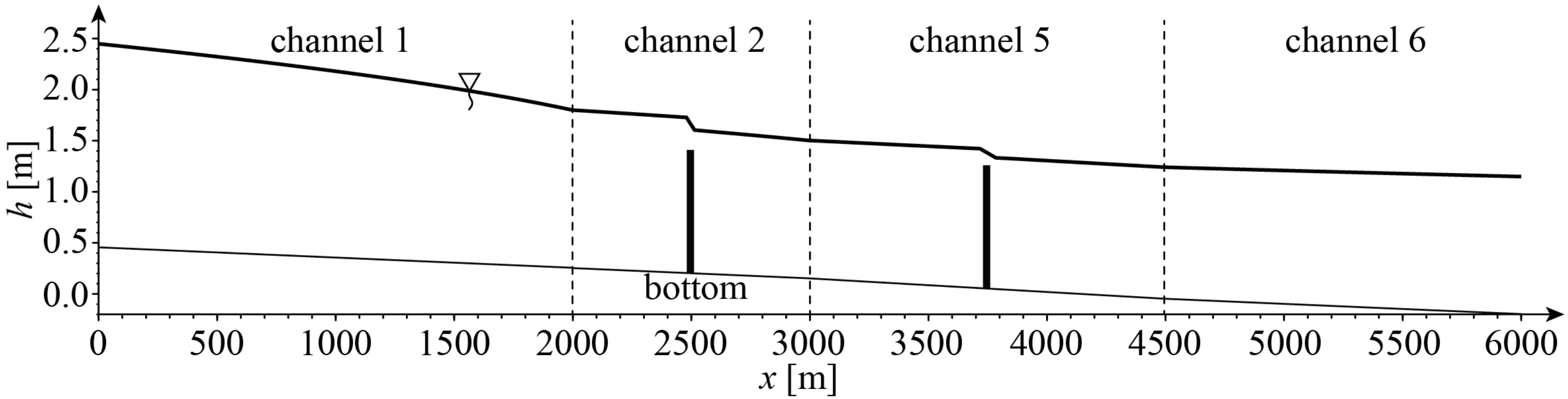
$$B_{orf} = 2.5\ \mathrm{m}$$

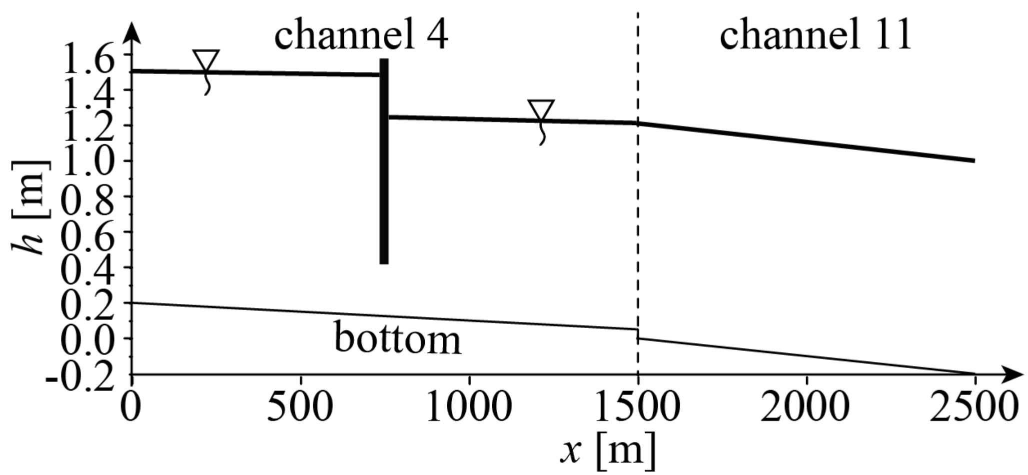
$$D_{orf} = 0.3\ \mathrm{m}$$

$$\mu_{orf} = 0.67$$

Computations outcome

No.	Q [m^3/s]
1	5.502
2	2.751
3	2.751
4	1.087
5	2.207
6	1.477
7	2.207
8	1.477
9	0.730
10	0.730
11	2.547


channel 1
channel 2
channel 5
channel 6
2.5
2.0
1.5
1.0
0.5
0.0
h [m]
bottom
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
5500
6000
x [m]


channel 4
channel 11
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
h [m]
bottom
0
500
1000
1500
2000
2500
x [m]

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HISTORY IS WISDOM
FUTURE IS CHALLENGE